16.06 Principles of Automatic Control
Lecture 16

Frequency Response Design

Problems with root locus:

- Works only for rational transfer functions:

$$G(s) = \frac{\pi(s - z_i)}{\pi(s - p_i)}$$

If we start with experiment data, it may be difficult or impossible to put $G(s)$ in the form above.

- The connection between performance of the system and the root locus is sometimes poor, so it is difficult to design for some kind of performance.

- For simple problems, can guess that right solution is lead (for greater stability or faster response) or lag (for greater position or velocity constant). For other problems, it's harder to guess the right form of the compensator.

Frequency response methods consider the behavior of $G(s)$, $K(s)$ along the $j\omega$ axis. That is, we look only at plots of $G(j\omega)$, $K(j\omega)$ to determine stability, performance.

Some advantages:

- Works with any $G(s)$ (or $G(j\omega)$), whether rational or not.

- It's relatively easy to determine $G(j\omega)$ from experimental data.

- Performance requirements can usually be specified in terms of the frequency response characteristics.

- The stability test is straightforward.

- There are a few simple rules for determining the type of compensator to use.

1

One disadvantage:

- Some of the *ideas* are difficult, even if the *techniques* are not.

The Frequency Response

Consider an LTI system G with input u and output y:

$$u \longrightarrow \boxed{G} \longrightarrow y$$
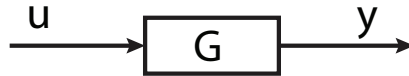
If the input is $u(t) = e^{st}$, then a particular solution is:

$$y(t) = G(s)e^{st}$$

For a sinusoidal signal

$$
\begin{aligned}
u(t) &= A \cos \omega t \\
&= A \mathrm{Re}(e^{j\omega t})
\end{aligned}
$$

the response is

$$
\begin{aligned}
y(t) &= A \mathrm{Re}(G(j\omega)e^{j\omega t}) \\
&= A(\mathrm{Re}[G(j\omega)] \cos \omega t - \mathrm{Im}[G(j\omega)] \sin \omega t)
\end{aligned}
$$

So the response is sinusoidal, with amplitude AM, where

$$
\begin{aligned}
M &= |G(j\omega)| \\
&= \sqrt{(\mathrm{Re}[G(j\omega)])^2 + \mathrm{Im}[G(j\omega)]^2}
\end{aligned}
$$

$y(t)$ can be expressed more simply as

$$y(t) = AM \cos(\omega t + \phi)$$

where

$$
\begin{aligned}
\phi &= \tan^{-1}\left(\frac{\mathrm{Im}[G(j\omega)]}{\mathrm{Re}[G(j\omega)]}\right) \\
&\equiv \angle G(j\omega)
\end{aligned}
$$

G may be expressed in polar form as

$$G(j\omega) = Me^{j\phi}$$
$$= M\angle\phi$$

So if we plot $|G(j\omega)|$ and $\angle G(j\omega)$ versus ω, we can easily determine the response to any sinusoidal input.

Terminology:

$M = |G(j\omega)| = $ magnitude of $G(j\omega)$

$\phi = \angle G(j\omega) = $ phase of $G(j\omega)$

Frequency Response of Lead Compensator

Recall that lead compensator has the form:

$$K(s) = k\frac{s + \zeta}{s + p}, \ p > z$$
$$\uparrow$$
$$\text{R.L. gain}$$

Rewrite this as

$$K(s) = \frac{k\zeta}{p}\frac{1 + s/\zeta}{1 + s/p}$$
$$= K\frac{1 + s/\zeta}{1 + s/p}$$
$$\uparrow$$
$$\text{"Bode" gain}$$

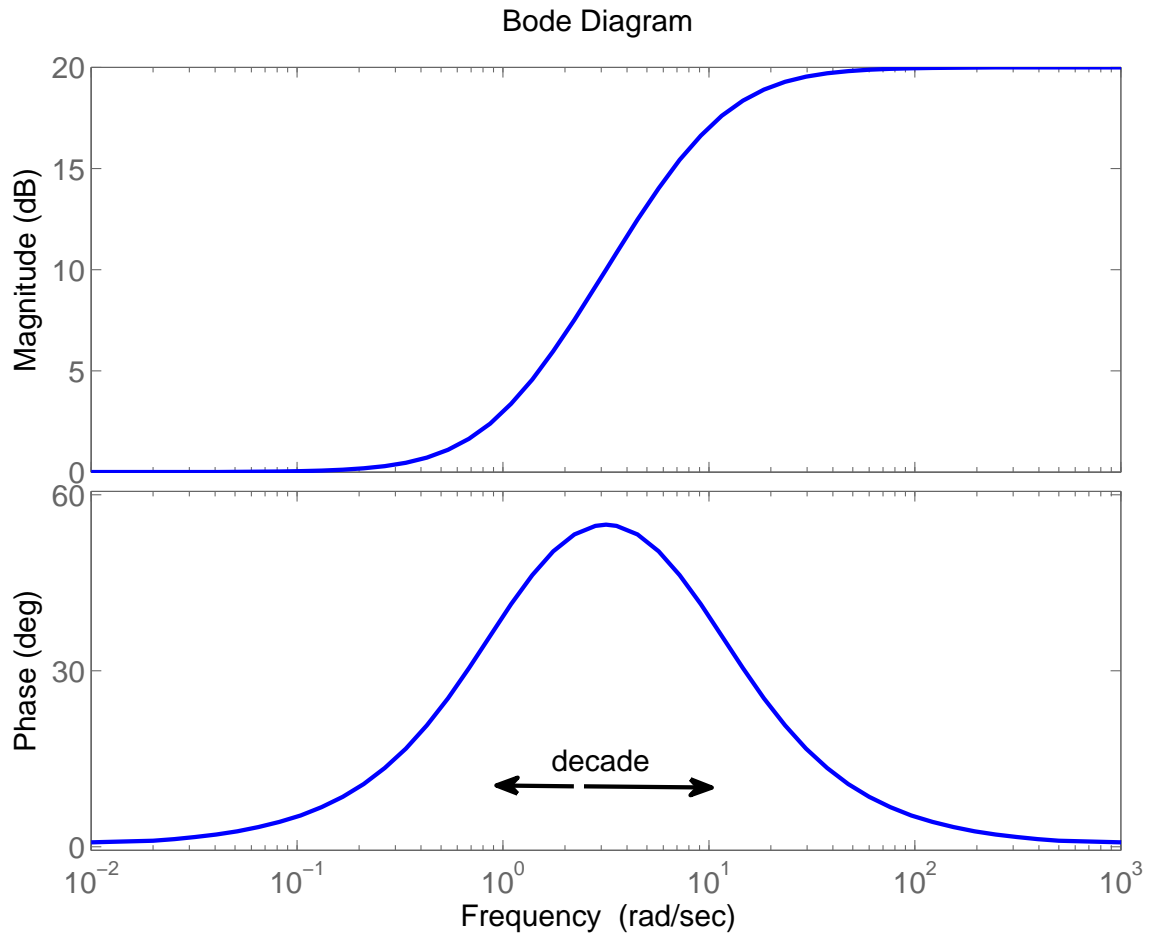What is magnitude and phase?

$$|G(j\omega)| = K\frac{\sqrt{1 + (\omega/\zeta)^2}}{\sqrt{1 + (\omega/p)^2}}$$
$$\angle G(j\omega) = \tan^{-1}(\omega/\zeta) - \tan^{-1}(\omega/p)$$

Traditionally, $|G|$ is plotted on a log-log plot, $\angle G$ on a semilog plot, together called *Bode plot*.

3

Example:

$$K(s) = \frac{1+s}{1+s/10}$$

Bode Diagram

Notes:

- Low-frequency magnitude $= 1 = $ K(0)

- High-frequency magnitude $=10=$K(∞)

- The phase is always positive - hence phase *lead*.

Often, magnitude is expressed in decibels, which is basically the log of $|G|$:
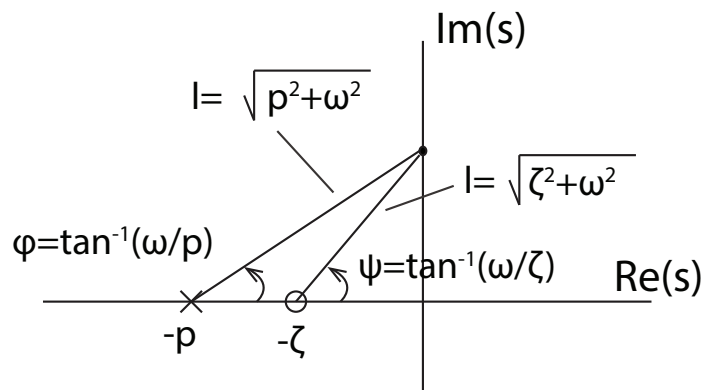
4

$$\text{magnitude in decibels} = 20\log_{10}|G|$$

Visualization using Root Locus plane

In root locus form, the lead compensator is

$$K(s) = \underbrace{\frac{1+s/\zeta}{1+s/p}}_{\text{Bode form}} = \underbrace{\frac{p}{\zeta}\frac{s+\zeta}{s+p}}_{\text{root locus form}}$$

Bode gain=1   R.L. gain=$\dfrac{p}{\zeta}$

So $|G| = \dfrac{p}{\zeta}\dfrac{\sqrt{\zeta^2+\omega^2}}{\sqrt{p^2+\omega^2}} = \dfrac{\sqrt{1+(\omega/\zeta)^2}}{\sqrt{1+(\omega/p)^2}}$, as before.

$$\angle G = \Psi - \phi$$
$$= \tan^{-1}\omega/\zeta - \tan^{-1}\omega/p, \text{ as before.}$$

16.06 Principles of Automatic Control
Fall 2012