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16.07 Dynamics
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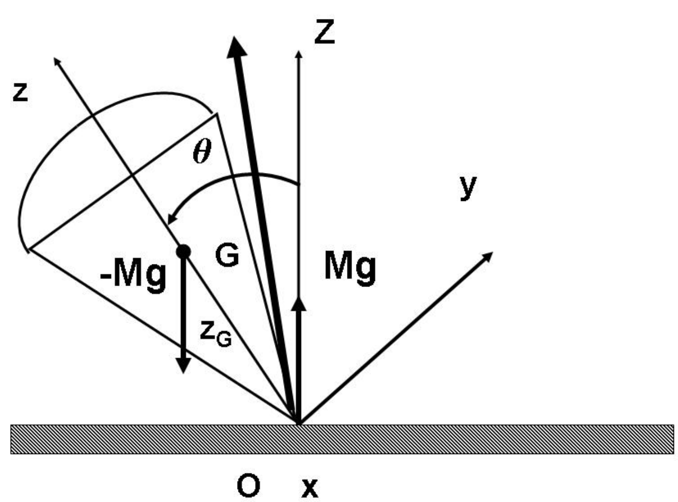
Lecture L30 - 3D Rigid Body Dynamics: Tops and Gyroscopes

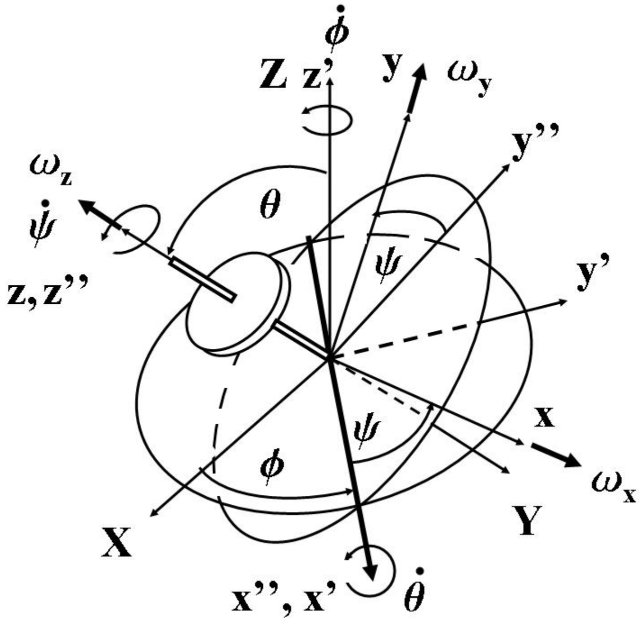
3D Rigid Body Dynamics: Euler Equations in Euler Angles

In lecture 29, we introduced the Euler angles as a framework for formulating and solving the equations for conservation of angular momentum. We applied this framework to the free-body motion of a symmetrical body whose angular momentum vector was not aligned with a principal axis. The angular moment was however constant. We now apply Euler angles and Euler's equations to a slightly more general case, a top or gyroscope in the presence of gravity.

We consider a top rotating about a fixed point O on a flat plane in the presence of gravity. Unlike our previous example of free-body motion, the angular momentum vector is not aligned with the Z axis, but precesses about the Z axis due to the applied moment. Whether we take the origin at the center of mass G or the fixed point O, the applied moment about the x axis is $M_x = Mgz_G sin\theta$, where z_G is the distance to the center of mass..



Initially, we shall not assume steady motion, but will develop Euler's equations in the Euler angle variables ψ (spin), ϕ (precession) and θ (nutation).



Referring to the figure showing the Euler angles, and referring to our study of free-body motion, we have the following relationships between the angular velocities along the x, y, z axes and the time rate of change of the Euler angles. The angular velocity vectors for $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$ are shown in the figure. Note that these three angular velocity vectors are not orthogonal, giving rise to some cross products when the angular velocities ω_i are calculated about the three principal axes.

$$\omega_x = \dot{\phi} \sin\theta sin\phi + \dot{\theta}\cos\psi \tag{1}$$

$$\omega_y = \dot{\phi} sin\theta \cos\phi - \dot{\theta}\sin\psi \tag{2}$$

$$\omega_z = \dot{\phi} cos\theta + \dot{\psi} \tag{3}$$

3D Rigid Body Dynamics: Euler's Equations

We consider a symmetric body, appropriate for a top, for which the moments of inertia $I_{xx} = I_{yy} = I_0$ and $I_{zz} = I$. The angular momentum is then

$$H_x = I_0\omega_x \tag{4}$$

$$H_y = I_0\omega_y \tag{5}$$

$$H_z = I\omega_z \tag{6}$$

For the general motion of a three-dimensional body, we have Euler's equations in ***body-fixed axes*** which rotate with the body so that the moment of inertia is constant in time. In this body-fixed coordinate system, the conservation of angular momentum is

$$\dot{\boldsymbol{H}} = \frac{d}{dt}\left([I]\{\omega\}\right) = AppliedMoments \tag{7}$$

Since we have chosen to work in a rotating coordinate system so that $\frac{d}{dt}I = 0$, we must pay the price, applying Coriolis theorem to obtain the time derivative of the angular velocity vector in the rotating coordinate system

$$\dot{\boldsymbol{H}} = \frac{d}{dt}\boldsymbol{H} + \Omega \times \boldsymbol{H}, \tag{8}$$

resulting in the "Euler" equations expressed in the x, y, z coordinate system moving with the body. In general, we must rotate with the total angular velocity $\boldsymbol{\omega}$ of the body, so that the governing equation for the conservation of angular momentum become, with $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$M_x = \dot{H}_x - H_y\Omega_z + H_z\Omega_y \tag{9}$$

$$M_y = \dot{H}_y - H_z\Omega_x + H_x\Omega_z \tag{10}$$

$$M_z = \dot{H}_z - H_x\Omega_y + H_y\Omega_x \tag{11}$$

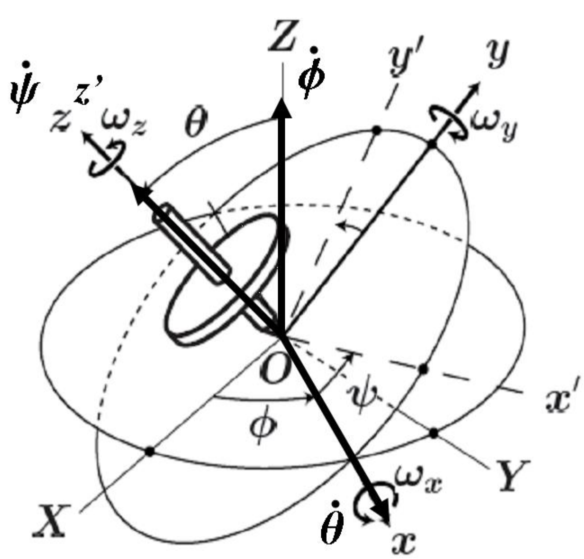
where Ω is the rotational angular velocity of our axis system. In this case because of the symmetry of the body, we are free to **choose** to allow the spin velocity to rotate relative to our body-fixed axes system and fasten our axis system to $\dot{\phi}$ and $\dot{\theta}$. That is, we follow the motion of the top in θ and ϕ but allow it to spin with angular velocity $\dot{\psi}$ relative to our coordinate system rotating with angular velocity $\boldsymbol{\Omega}$. So that

$$\Omega_x = \dot{\theta} \tag{12}$$

$$\Omega_y = \dot{\phi} sin\theta \tag{13}$$

$$\Omega_z = \dot{\phi} cos\theta. \tag{14}$$

For this choice of coordinate system, we have essentially performed a rotation in ϕ and θ only, leading to the geometry shown.



In this coordinate system, since the ψ rotation did not occur, the angular velocity of the body is

$$\omega_x = \dot{\theta} \tag{15}$$

$$\omega_y = \dot{\phi} sin\theta \tag{16}$$

$$\omega_z = \dot{\phi} cos\theta + \dot{\psi}. \tag{17}$$

For this choice of coordinate system, we have

$$H_x = I_0\dot{\theta} \tag{18}$$

$$H_y = I_0\dot{\phi}\sin\theta \tag{19}$$

$$H_z = I(\dot{\phi}\cos\theta + \dot{\psi}) \tag{20}$$

resulting in

$$\dot{H}_x = I_0\ddot{\theta} \tag{21}$$

$$\dot{H}_y = I_0(\ddot{\phi} sin\theta + \dot{\phi}\dot{\theta}\cos\theta \tag{22}$$

$$\dot{H}_z = I(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}) \tag{23}$$

and $\boldsymbol{\Omega} \times \boldsymbol{H}$ in components is

$$(I - I_0)\cos\theta\sin\theta\dot{\phi}^2 + I\sin\theta\dot{\phi}\dot{\psi} \tag{24}$$

$$(I_0 - I)\cos\theta\dot{\phi}\dot{\theta} - I\dot{\psi}\dot{\theta} \tag{25}$$

$$0. \tag{26}$$

Note that the z component is zero. Since the axes chosen are principal axes, the final form of Euler's equations becomes,

$$M_x = I_0(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}) \tag{27}$$

$$M_y = I_0(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \tag{28}$$

$$M_z = I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) \tag{29}$$

For a top, $M_x = Mgz_G sin\theta$, $M_y = 0$ and $M_z = 0$. These equations are unsteady and non-linear. We can gain insight by examining the character of some special solutions and constants of the motion.

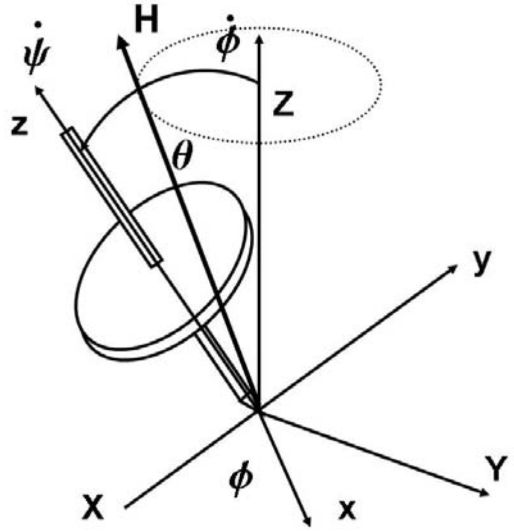
Steady Precession: Gyroscopic Motion

We now consider the steady precession of a top about the Z axis. In terms of the variables we have defined, the top rotates with spin velocity $\dot{\psi}$ about its principal axis, and precesses with angular velocity $\dot{\phi}$ while maintaining a constant angle θ with the vertical axis Z. We then have

$$\dot{\phi} = constant = \dot{\phi}_0 \qquad \ddot{\phi} = 0 \tag{30}$$

$$\theta = constant = \theta_0 \quad \dot{\theta} = \ddot{\theta} = 0 \tag{31}$$

$$\dot{\psi} = 0 \qquad \ddot{\psi} = 0 \tag{32}$$



For a steady precession of a top, Euler's equations reduce to

$$\dot{\phi} sin\theta (I(\dot{\phi} cos\theta + \dot{\psi}) - I_0 \dot{\phi} cos\theta) = M_x = Mg sin\theta z_G \tag{33}$$

Note that $sin\theta$ cancels, resulting in

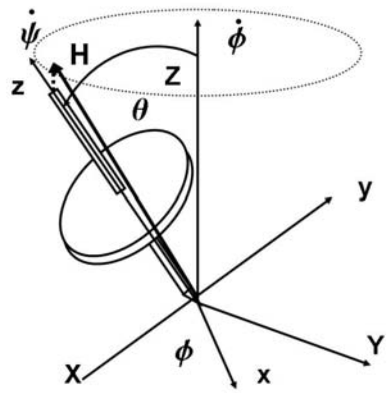
$$I\dot{\phi}\dot{\psi} - (I_0 - I)\dot{\phi}^2 cos\theta = Mgz_G \tag{34}$$

In the usual case for tops or gyroscopes, we have $\dot{\psi} >> \dot{\phi}$ so that $\dot{\phi}^2$ may be ignored. Therefore, for steady precession, the relationship between the precession angular velocity and the spin angular velocity is

$$\dot{\phi} = Mgz_G/(I\dot{\psi}) \tag{35}$$

where $I = I_{zz}$, the moment of inertia of the gyroscope about its spin axis. This result is also true if $\theta = 0$. For this motion, the angular momentum vector is not aligned with the Z axis as for free-body motion, but is in the plane of z, Z, and rotates around the Z axis according to the applied external moment which is constant and in the x direction. In the limit as $\dot{\psi} >> \dot{\phi}$ the angular momentum vector is **essentially** along the axis of rotation, z, with a **slight** component due to $\dot{\phi}$ which can be calculated **after** $\dot{\psi}$ has been determined. Some attempt to sketch this is shown in the figure, but even this small correction is an approximation for $\dot{\psi} >> \dot{\phi}$. For $\dot{\psi} >> \dot{\phi}$, this same result can be obtained by less complex approach, but it is important to realize that this is an approximation. For the spin velocity much greater than the precessional velocity, we may take the angular momentum vector as directed **along** the spin or z axis, $\boldsymbol{H} = I\omega\mathbf{k}$. Then the applied moment will precess the vector with rotation rate $\boldsymbol{\Omega} \times \boldsymbol{H}$. Since $\boldsymbol{\Omega} = \Omega\mathbf{K} = \dot{\phi}\mathbf{K}$, (where $\mathbf{K}$ is a unit vector in the Z direction), this give a result for Ω in agreement with the previous result

$$\Omega = Mgz_G/(I\omega) \tag{36}$$



Since θ dropped out of the previous equation, for $\theta = 0$, we again have $\Omega = Mgz_G/(I\omega)$.
The limit $\dot{\psi} >> \dot{\phi}$ is the "gyroscopic" limit where the device behaves as a gyroscope rather than as the more general case of a top. The difference is that, for a gyroscope, ω is larger than any other rotation rate in the system, such as the angular velocity of an aircraft or spacecraft. This makes the gyroscope a useful basis for many instruments. We shall return to this issue.

Unsteady Precession of a Top: Integrals of Motion

For the general case of a top in a gravitational field, we have Euler's Equations. In the most general case, we will have spin $\dot{\psi}$, precession $\dot{\phi}$ and **nutation** $\dot{\theta}$, all varying with time.

$$\begin{aligned} M_x &= I_0(\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\phi}cos\theta + \dot{\psi}) = Mgz_G\sin\theta & (37)\\ M_y &= I_0(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) = 0 & (38)\\ M_z &= I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) = 0 & (39)\end{aligned}$$

The consequences of $M_y = 0$ and $M_z = 0$ provide two quantities that must be constant/conserved during the motion. (We will later show that these results can also be obtained by applying Lagrange's equation to this system.)

$$\begin{aligned} d/dt(I_0\sin^2\theta\dot{\phi} + I\cos\theta(\dot{\psi} + \dot{\phi}cos\theta)) = & \ \tfrac{dp_\phi}{dt} = 0 & (40)\\ d/dt(I(\dot{\psi} + \dot{\phi}\cos\theta)) = & \ \tfrac{dp_\psi}{dt} = 0 & (41)\end{aligned}$$

It can easily be seen by inspection that differentiating p_ψ and setting it equal to 0 yields the equation $M_z = 0$. The equation from $\frac{dp_\phi}{dt}$ is more complex; to obtain M_y requires combinations of $\frac{dp_\phi}{dt}$ and $\frac{dp_\psi}{dt}$.

But the conclusion is that p_ψ and p_ψ are constants of the motion; we will show later that these can be considered generalized momenta, derived by application of Lagrange's equation.
Therefore, the spin angular velocity and the precession angular velocity are instantaneously related to the nutation angle θ through

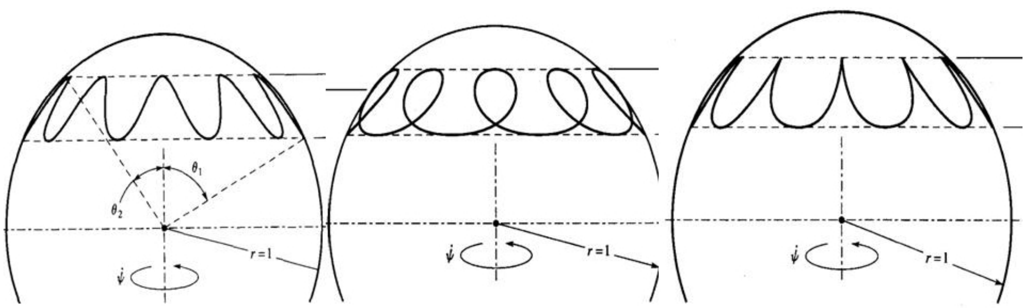
$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_0 \sin^2\theta} \tag{42}$$

$$\dot{\psi} = \frac{p_\psi}{I_0} - \frac{p_\phi - p_\psi \cos\theta}{I sin^2\theta} \tag{43}$$

We may therefore write the remaining governing equation as

$$\ddot{\theta} + (\frac{\frac{p_\phi}{I_0} - \frac{p_\psi}{I} cos\theta)(\frac{p_\psi}{I} - \frac{p_\phi \cos\theta}{I_0})}{sin^3\theta} - \frac{gMz_G}{I_0} = 0 \tag{44}$$

This reduces the problem to the motion of a single variable θ; once $\theta(t)$ has been determined, the constancy of p_ϕ/I_0 and p_ψ/I during the motion, give the relations between the nutation angle $\theta(t)$, the spin angle $\psi(t)$, and precession angle $\phi(t)$. For this more complex motion, all three angles change with time, and the tip of the top traces out a motion, inscribed on the surface of a sphere for visualization, as shown in the figure. These various motions are referred to as unidirectional precession, looping precession and cuspidal motion.



unidirectional precession **looping precession** **cuspidial precession**

For more discussion of these solutions see J.B Marion and S.T. Thornton, *Classical Dynamics*, Chapter 11.

Spinning Top by Lagrange's Equation

The constancy of two momenta obtained by application of Euler's equation can be found perhaps more directly by application of Lagrange's equation. We write the kinetic energy of a spinning top as

$$T = 1/2 I_0(\omega_x^2 + \omega_y^2) + 1/2 I\omega_z^2 \tag{45}$$

We use the full form of the angular velocities in the Euler system moving with the spinning top and obtain

$$\omega_x^2 = (\dot{\phi} \sin\theta sin\phi + \dot{\theta}\cos\psi)^2 \tag{46}$$

$$\omega_y^2 = (\dot{\phi} sin\theta \cos\phi - \dot{\theta}\sin\phi)^2 \tag{47}$$

$$\omega_z^2 = (\dot{\phi} cos\theta + \dot{\psi})^2 \tag{48}$$

resulting in

$$\omega_x^2 + \omega_y^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \tag{49}$$

$$\omega_z^2 = (\dot{\phi}\cos\theta + \dot{\psi})^2. \tag{50}$$

Therefore the kinetic energy is

$$T = 1/2 I_0(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + 1/2 I(\dot{\phi}\cos\theta + \dot{\psi})^2 \tag{51}$$

For a top, the potential energy is

$$V = Mgz_G \cos\theta \tag{52}$$

so that the Lagrangian is

$$L = 1/2 I_0(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + 1/2 I(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgz_G\cos\theta. \tag{53}$$

We consider θ, ϕ and ψ to be our generalized coordinates, q_i. And we move forward applying Lagrange's equation, $d/dt(\partial L/\partial \dot{q}_i) - \partial V/\partial q_i = 0$. However, since $\partial V/\partial q_i = 0$ for both $q_i = \phi$ and $q_i = \psi$, we conclude that

$$\frac{\partial L}{\partial \dot{\phi}} = (I_0 \sin^2\theta + I\cos^2\theta)\dot{\phi} + I\dot{\psi}\cos\theta = constant = p_\phi \tag{54}$$

$$\frac{\partial L}{\partial \dot{\psi}} = I(\dot{\psi} + \dot{\phi}\cos\theta) = constant = p_\phi \tag{55}$$

We define these groupings which must be constant in time as generalized momenta, consistent with our earlier exposition of Lagrange's equations. These are of course the same constants identified using Euler's equations in equations (38-39). Lagrange's equations win! The final governing equation is obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{56}$$

resulting in

$$\ddot{\theta} + (\frac{\frac{p_\phi}{I_0} - \frac{p_\psi}{I}cos\theta)(\frac{p_\psi}{I} - \frac{p_\phi\cos\theta}{I_0})}{sin^3\theta} - \frac{gMz_G}{I_0} = 0, \tag{57}$$

in agreement with equation (44).

We also observe that total energy is a constant of the motion.

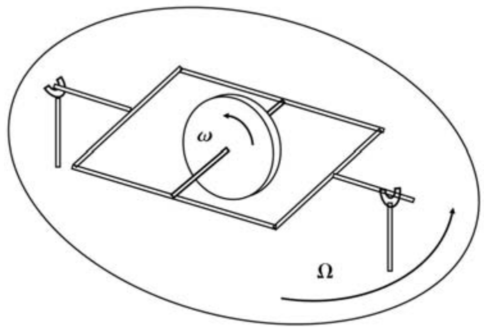
$$E = 1/2 * I_0(\dot{\phi}^2\sin^2\theta + theta^2) + 1/2 I_0\dot{\phi}\cos\theta + \dot{\psi})^2 + MHz_G\cos\theta \tag{58}$$

For more discussion of these solutions see J.B Marion and S.T. Thornton, *Classical Dynamics*, Chapter 11.

Extreme Tops: Gyroscopes

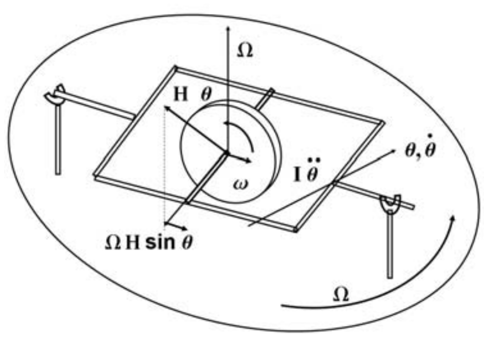
Considerable simplification and practical application results if the angular velocity ω is an order of magnitude larger than angular velocity associated with the motions of a system of interest, such as an aircraft or a spacecraft. In this case, the gyroscope can be used as an instrument to measure some quantity of interest in a vehicle.

In order to use a gyroscope as an instrument, we first have to consider how it is mounted and affixed to the vehicle. The classic mount is called a gimbal. It permits free rotation of the gyroscope about its own axis, and depending on how complex the gimbal system is, free rotation about other axis as well.



Consider the gyroscope shown in the figure. It is free to spin about its axis with angular velocity ω. It is attached to a gimbal which is free to rotate about the support as sketched. The entire apparatus is mounted on a turntable rotating with angular velocity Ω, where $\omega >> \Omega$. If we can measure the orientation of the axis of rotation of the disk relative to the turntable mount, we could possibly use this as an instrument. What does it measure?

Consider that the initial angular momentum vector of the disk is in the horizontal plane. The axis of the disk is free to rotate about the horizontal axis through an angle θ. As the turntable rotates with angular velocity Ω, the axis of the disk will rotate in θ–its only free direction– to keep its rate of change of angular momentum zero, since it can experience no moments/torques. There are two source of changes in angular momentum, the unsteady component if θ changes with time; and the component of the angular momentum of the disk in the horizontal plane that is precessed by the turntable rotation Ω.



The unsteady term is $I_0\ddot{\theta}$, where I_0 is the moment of inertia of the disk about the axis perpendicular to its spin axis; the vector is in the horizontal direction. The horizontal component of the angular momentum of the disk is $I\omega sin\theta$, where I is the moment of inertia of the disk about its axis of rotation. The rate of change is also in the horizonal direction, perpendicular to Ω and of magnitude $I\omega\Omega sin\theta$. The sum of these two terms, which must be zero, is

$$I_0\ddot{\theta} + I\omega\Omega sin\theta = 0 \tag{59}$$

This is the familiar equation for the oscillation of a pendulum. If there is a small amount of friction in the support, the system will settle at $\theta = 0$ and will indicate the direction of rotation of the turntable axis. This is perhaps not too interesting, since we already know the direction of rotation of the turntable. But if we have this device on a rotating body, such as the earth, it will again indicate the direction of the axis of rotation of the body, in this case the earth. We call this direction "North". Thus, this application of the gyroscope could be used as a compass, often called a gyrocompass.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition
7/9

J.B Marion and S.T. Thornton, *Classical Dynamics*, Chapter 11

W.T. Thompson, *Introduction to Space Dynamics, Chapter 5*

J. H. Ginsberg, Advanced Engineering Dynamics*, Second Edition, Chapter 8*

D. Kleppner, R.J. Kolenkow, An introduction to Mechanics,Chapter 7

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