Turbulent Shear Layers.

7.4) Turbulence Modeling and Closure

A) Intro
B) Algebraic
C) Transport

Recap: Last lecture we looked at $\mathcal{G}$, H^*, C_D for turbulent flows. Derived two C_D

(1) $$C_D = \frac{\mathcal{G}}{2} U_s + \frac{\pi}{4} C_\tau (1-U_s)$$

$$F \cdot C_\tau (1-U_s)$$

(2) $$\frac{2C_D}{H^*} = \frac{\mathcal{G}}{2} U_s + 0.03 \left(\frac{H-1}{H}\right)^2 \frac{3}{4} (1-U_s)$$

$$\frac{2C_D}{H^*} = \mathcal{G}/2 \left[1 - \frac{1}{0.75} \frac{H-1}{H}\right) + 0.03 \left(\frac{H-1}{H}\right)^3$$

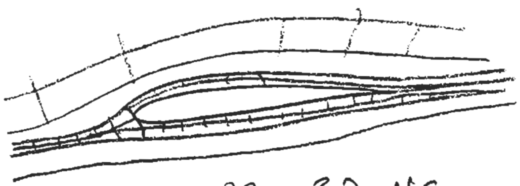
Alt expr for $U_s = \frac{H^*}{2}\left(1 - \frac{4}{3} \frac{H-1}{H}\right)$ (Drela AIAA Paper 86-1786

and $C_\tau = \frac{H^*}{2} \cdot \frac{0.03}{1-U_s} (\ )^3$

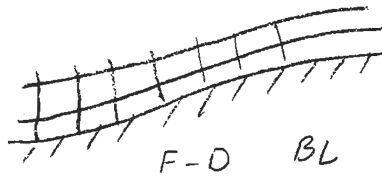
$$\Rightarrow \therefore \frac{2C_D}{H^*} = \frac{\mathcal{G}}{2} \cdot \frac{2U_s}{H^*} + C_\tau \frac{2}{H^*} (1-U_s)$$

$$\therefore C_D = \mathcal{G}/2 \cdot U_s + C_\tau (1-U_s)$$ (1 & 2 equiv)

In order to solve N-S Eqns in 2D or 3D, or simply calculate turbulent BL using F-D method we need turbulence models.



2D, 3D NS



F-D BL calc

Momentum Eqn.

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \cdot \overline{\overline{\tau}}_L + \nabla \cdot \overline{\overline{\tau}}_t$$

Recall

$$\rho \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{\partial p}{\partial x} + \mu \nabla^2 \bar{u}$$

$$- \frac{\partial}{\partial x}\left(\rho \overline{u'^2}\right) - \frac{\partial}{\partial y}\left(\rho \overline{u'v'}\right) - \frac{\partial}{\partial z}\left(\rho \overline{u'w'}\right)$$

$$\overline{\overline{\tau}}_L = 2\mu\left[\overline{\overline{e}} - 1/3(\nabla \cdot \vec{u})\,\overline{\overline{I}}\right] \qquad \overline{\overline{e}} = \nabla \vec{u}$$ (strain rate tensor

$$\overline{\overline{\tau}}_t = -\rho \begin{bmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \cdot & \overline{v'^2} & \overline{v'w'} \\ \cdot & \cdot & \overline{w'^2} \end{bmatrix} = -\rho \overline{u'_i u'_j}$$

Trace — $\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = 2 \cdot K$ (turbulence K.E)

Reynolds stresses are additional unknowns which have to related to $\vec{u}$, p, ... etc.

Two basic approaches.

① Eddy viscosity closure: assume $\overline{\overline{\tau}}_t = 2\mu_t \overline{\overline{e}} - 2K\overline{\overline{I}}$

- Unknowns are $\mu_t(x,y,z,t)$ and $K(x,y,z,t)$
- Typical imp terms $-\overline{u'v'}$ in 2D, $-\overline{u'v'}$, $\overline{v'w'}$ in 3D.
- Good assumption for regular aero flows, poor prediction if normal stresses are significant.

2) Reynold's stress closure: $\bar{\bar{\tau}}_t = -\rho\overline{u_i' u_j'}$ - directly model stress terms. $\overline{u'v'}(x,y,z,t)$, $\overline{u'^2}(x,y,z,t)$ ....

. 6 unknowns vs. 2 for ①

Two basic solution approaches:

① Algebraic Methods: explicit formulas for μ_t or $\overline{u'v'}$ in terms of mean flow $\vec{\bar{u}}(x,y,z,t)$

② Transport Methods: $\frac{D\mu_t}{Dt} = ...$

$$\frac{D\,\overline{u'v'}}{Dt} = ...$$

	Eddy	Stress
Alg	$\mu_t = f(\rho, \vec{\bar{u}})$ Ex: $\mu_t = k\rho u_e \delta^*$	$-\overline{u_i' u_j'} = f_{ij}(\rho, \vec{\bar{u}})$
Transp.	$\frac{D\mu_t}{Dt} = f(\rho, \vec{\bar{u}}, \mu_t)$ $k-\epsilon$ model	$\frac{D\,\overline{u_i' u_j'}}{Dt} = f_{ij}(\rho, \vec{\bar{u}}, \overline{u_i' u_j'})$ stress models.

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① 0 Eqn. or Algebraic Model

$$\mu_t \approx \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad A = \text{damping factor } (\beta)$$

$$l \approx ky\left[1 - e^{-y^+/A}\right] \qquad (\text{composite func})$$

· iteration required since there is dependance on τ_w, δ, δ^* etc.

Baldwin Lomax model - simple, popular for aerodynamic flows.

$$\mu_{t\,outer} \approx \frac{0.016\, C_{cp}\, \rho\, y_{max} F_{max}}{1 + 5.5\left(C_{kleb}\, y / y_{max}\right)^6}$$

$$F_{max} = m \left[y \left| \frac{\partial \bar{u}}{\partial y} \right| \left(1 - e^{-y^+/A}\right) \right]$$

$$C_{kleb} = 2/3 - f(\beta) \quad , \quad C_{cp} \approx \frac{3 - 4 C_{kleb}}{2 C_{kleb}\left(2 - 3 C_{kleb} + C_{kleb}^3\right)}$$

Alg. Stress Model $\overline{u_i' u_j'} = 2/3 K \delta_{ij}$ +

~~(2) One Equation (Transport) Methods~~

$$\frac{(1 - C_\gamma)(K/\epsilon)(P_{ij} - 1/3 P_{ij}\delta_{ij})}{C_1 + (1/2\,\epsilon) P_{ii} - 1}$$

P_{ij} = production $= -\overline{u_i' u_k'} \frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_k}$

K = kinetic energy

$\epsilon = (\text{const}) \dfrac{K^{3/2}}{L}$

(2) One Eqn Method :

- Conservation relation for turbulence K.E.

$$\frac{DK}{Dt} = \underbrace{\text{convection}}_{} \overset{\text{II}}{} \;(\text{diffusion}) + \overset{\text{III}}{\underline{\text{production}}} + \overset{\text{IV}}{\underline{\text{turb stress work}}} + \overset{\text{V}}{\underline{\text{dissipation}}}$$

<u>x direction</u>

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} \approx -\frac{\partial}{\partial y}\left[\overline{v'\left(1/2\, u_i' u_i' + p'/\rho\right)}\right] + \frac{\tau}{\rho}\frac{\partial \bar{u}}{\partial y} + \epsilon$$

Model terms on RHS:

$$\epsilon \approx (\text{const}) \frac{K^{3/2}}{L} \qquad (\text{dimensional argument})$$

$$-\overline{v'\left(1/2\, u_i' u_j' + p'/\rho\right)} \approx (\text{const}) \frac{\partial K}{\partial y}$$

$$\text{RHS} \approx \frac{\partial}{\partial y}\left(\text{const}\frac{\partial K}{\partial y}\right) + \nu_t \left(\frac{\partial u}{\partial y}\right)^2 - \text{const}\, K^{3/2}/L$$

+ algebraic relation for L

Spalart - Allmaras Model (well known, popular for aerodynamic flows)

$$\frac{D\nu_t}{Dt} = \underset{\uparrow\ \text{production}}{C_{b1}\nu_t S} + \underset{\text{const}}{\frac{1}{\sigma}}\left[\underset{\uparrow\ \text{diffusion}}{\frac{\partial}{\partial y}\left(\nu_t\frac{\partial \nu_t}{\partial y}\right)} + C_{b2}\left(\frac{\partial \nu_t}{\partial y}\right)^2\right] - \underset{\uparrow\ \text{destruction}}{C_{w1}\left(\frac{\nu_t}{y}\right)^2}$$

S - strain rate measure = ω vorticity

$= \|\bar{\bar{e}}\|$

2- Eqn Model. $\nu_t = \dfrac{C_\mu K^2}{\epsilon}$, solve

Energy : $$\frac{DK}{Dt} \approx \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_K}\frac{\partial K}{\partial x_j}\right) + \nu_t\frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \epsilon$$

Dissip : $$\frac{D\epsilon}{Dt} \approx \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) + C_1\nu_t\frac{\epsilon}{K}\ \text{''} \ - C_2\frac{\epsilon^2}{K}.$$

C_μ, σ_K, σ_ϵ, C_1, C_2 constants of order unity.

Typically combined with wall functions (assume log law near wall). Less grid - computational saving.

Reynolds Stress Model

$$\frac{D}{Dt}(\overline{u'v'}) = \underset{\uparrow\ \text{diff}}{D_{ij}} + \underset{\uparrow\ \text{prod}}{P_{ij}} + \underset{\uparrow\ \text{pressure strain}}{\Pi_{ij}} - \underset{\uparrow\ \text{dissipation}}{\epsilon_{ij}} + \nu\nabla^2(\overline{u'v'})$$