Lecture 7

3> Thin Shear Layer Approximation

2> A> $Re \rightarrow \infty$ behavior

B> Ordering

C> TSL Approximation

96 – 99

Sch - 145 – 148

White - 218 - 219

227 – 233.

Reading: White 218 - 219, 227 - 233

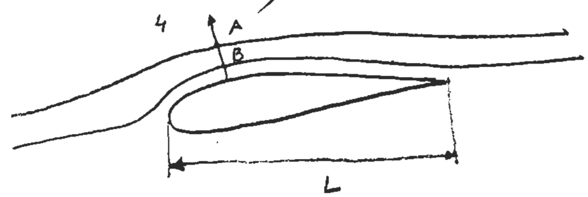
Sch. (see New ed.)

* ~~~

A>

$$\underset{1}{\frac{\partial u}{\partial t}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{2} = \underset{3}{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \underset{4}{\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)} \quad \text{— x comp.}$$

$$Re = \frac{u_\infty L}{\nu} >> 1$$



At A, 1, 2, and 3 balance

B, 1 & 2 $\rightarrow 0$ 3 ~ 0 (4)

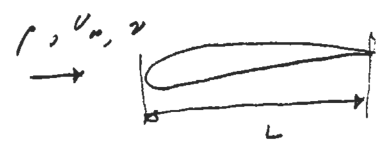
↑ do 1st

* Using ρ, U_∞, ν, L as scales, the governing equations are:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \frac{1}{Re}\nabla^2\vec{u}$$

$$Re \equiv \frac{U_\infty L}{\nu}$$



B.C $\vec{u} = 0$ at wall (no slip)

Typical Re values are large

Example

Pigeon	50K
Auto, Cessna	5 mill.
747	100 mill
Super tanker	5 bill

This suggests that $1/Re$ is a small parameter → seek solutions as an asymptotic expansion in ϵ $((1/Re)^p)$.

$$\vec{u} = \vec{u}_0 + \epsilon \vec{u}_1 + \epsilon^2 \vec{u}_2 + \cdots \qquad *$$
$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots$$

Look first at $\vec{u}_0$, p_0 : put (*) in N-S equs. and B-Cs.

$$\nabla \cdot \vec{u}_0 = 0$$
$$\frac{\partial \vec{u}_0}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_0 = -\nabla p_0$$
$$B\cdot C: \ \vec{u}_0 = 0$$

Problem: Cannot satisfy both $u_0 = 0$ & $V_0 = 0$ at wall

only $\vec{u}_0 \cdot \hat{n} = 0$

We lost highest-order term $\epsilon^2 \nabla^2 \vec{u}$

→ Singular perturbation problem.

No slip B·C forces $\epsilon^2 \nabla^2 \vec{u}$ to be finite as $\epsilon \to 0$

The "fix" is to seek scales other than U_∞, L near wall region

Example: In Rayleigh case, we had $\delta(t) = \sqrt{\nu t}$; $Y = y/\delta(t)$

In B·L case, look for $\delta(x)$ for scaling in lieu of L for y

In the above problem we can switch the y coordinate

$$Y = y/\epsilon \qquad s.t \quad Y = O(1) \ \text{as} \ \epsilon \to 0$$

Near the wall we can use

$$u = u_1 + \epsilon u_2 + \cdots$$
$$v = \epsilon v_1 + \epsilon^2 v_2 + \cdots$$
$$p = p_0 + \epsilon p_1 + \cdots$$

X comp $\Rightarrow$ $$u_1 \frac{\partial u_1}{\partial X} + v_1 \frac{\partial u_1}{\partial Y} = -\frac{\partial p}{\partial X} + \underbrace{\frac{1}{\epsilon^2 Re}} \cdot \frac{\partial^2 u_1}{\partial Y^2}$$

$$\to \epsilon = \frac{1}{\sqrt{Re}}$$

* Simple linear ODE illustrates ① loss of highest derivative ② choice of length scale near a wall.

$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = a \qquad f(0) = 0 \ , \quad f(1) = 1$$

Exact solution:

$$f(x;\epsilon) = (1-a)\left(\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}\right) + ax$$

First setting $\epsilon = 0$ gives

$$\frac{df}{dx} = a$$

which can only satisfy one boundary condition unless $a = 1$

$$f(x;\epsilon) \sim (1-a) + ax \quad \text{as} \quad \epsilon \to 0$$

(result of dropping highest derivative)

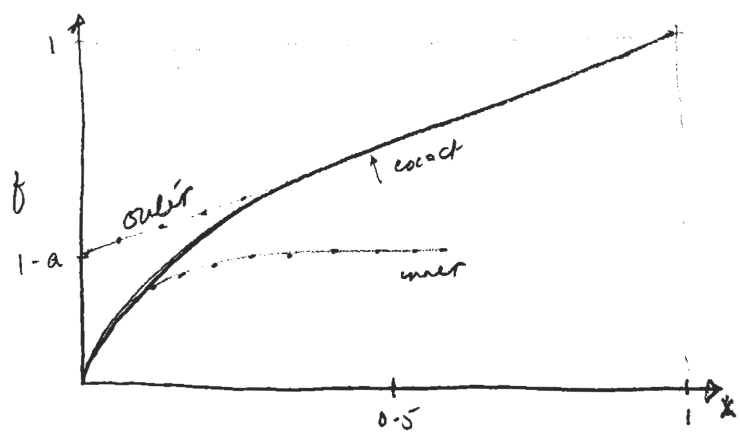
Choose a different scale when x is small or close to the wall

$$X = x/\epsilon \longrightarrow F(X;\epsilon)$$

Substituting gives

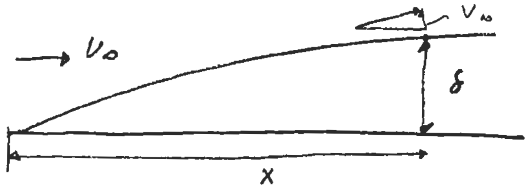
$$\frac{d^2 F}{dX^2} + \frac{dF}{dX} = a\epsilon \qquad F(0) = 0 \ , \quad F(1/\epsilon) = 1$$

$$\Rightarrow \quad f(x;\epsilon) \sim (1-a)(1 - e^{-x/\epsilon}) \qquad \text{as } \epsilon \to 0 \text{ but } X \sim O(1)$$



Ref: Van Dyke
Pert. Methods in Fluid Dynamics

B) Ordering



Examine the order of magnitude of each term in governing eqns.

1) continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$O\left\{ \frac{U_\infty}{x} \quad \frac{V_\infty}{\delta} \right\} \longrightarrow \frac{\delta}{x} = O\left(\frac{V_\infty}{U_\infty}\right)$$

$$O\left\{ 1 \quad 1 \right\} \qquad x = O(1)$$

2) X-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$O\left\{ \frac{U_\infty \cdot U_\infty}{x} \quad U_\infty\left(\frac{\delta}{x}\right)\frac{U_\infty}{\delta} \quad \frac{U_\infty^2}{x} \quad \delta^2\left[\frac{U_\infty}{x^2} \quad \frac{U_\infty}{\delta^2}\right] \right.$$

$$\frac{U_\infty^2}{x} \quad \frac{U_\infty^2}{x} \qquad \delta^2\left[\frac{U_\infty}{x^2} \quad \frac{U_\infty}{\delta^2}\right]$$

$$1 \quad 1/\delta^2$$

Also $\frac{U_\infty^2}{x} = O\left(\nu\frac{U_\infty}{\delta^2}\right) \Rightarrow \frac{\delta}{x} = O\left(\sqrt{\frac{\nu}{U_\infty x}}\right)$

$$\frac{\partial^2 u}{\partial x^2} \sim O(\delta^2)$$

3) Y momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

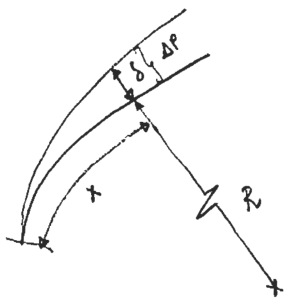
$$O\left\{ \frac{U_\infty \cdot U_\infty \delta/x}{x} \quad U_\infty\frac{\delta}{x} \cdot U_\infty \delta/x \, \frac{1}{\delta} \quad \delta^2\left[\frac{U_\infty \delta}{x^2 x} \quad \frac{U_\infty}{\delta^2}\cdot\frac{\delta}{x}\right] \right.$$

$$\delta \quad \delta \quad \delta^3 \quad \delta$$

This suggests

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = O\left(\frac{U_\infty^2}{X^2}\cdot\delta\right) \quad \text{or} \quad O(\delta)$$

Curved wall



$$\Rightarrow \quad \frac{1}{\rho}\frac{\partial p}{\partial y} = O\left(\frac{U_\infty^2}{R}\right)$$

Change in pressure across B-L

$$p(\delta) - p(0) = \Delta p \cong \frac{\partial p}{\partial y}\delta = O\left(\rho u^2 \left(\delta/x\right)^2\right) \quad \text{or} \quad O(\delta^2)$$

$$\text{or} \quad = O\left(\rho u^2 \left(\delta/R\right)\right) \quad \text{or} \quad O(\delta/R)$$

which is bigger in most cases.

In summary,

Keeping terms of $O(1)$ in x-momentum, and $\frac{\partial p}{\partial y} \approx 0$ in y-momentum gives us <u>TBL Equations</u>

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial p}{\partial y} = 0 \end{aligned}}$$

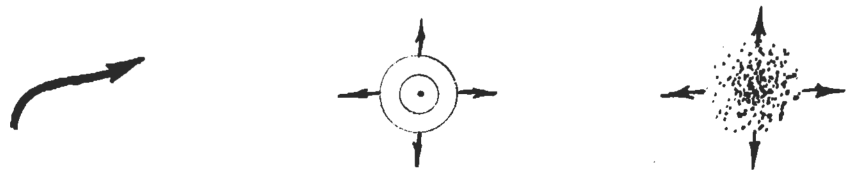
Neglects streamwise diffusion $\nu\frac{\partial^2 u}{\partial x^2} \approx 0$

Neglects normal momentum $\frac{\partial p}{\partial y} \approx 0$

* assumption weakest at airfoil t.e and shocks, for example

THIN SHEAR LAYER APPROXIMATION

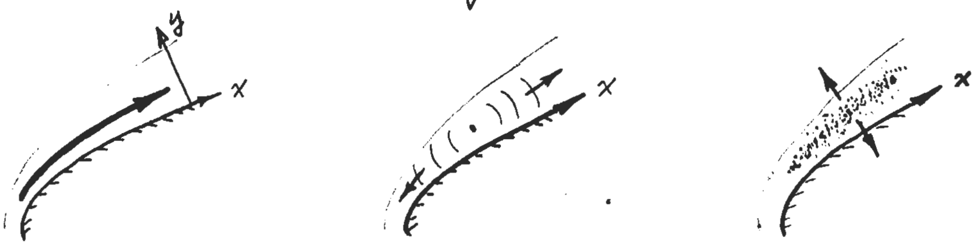
Viscous flows contain 3 basic momentum transport mechanisms:



CONVECTION PRESSURE DIFFUSION

$$\vec{u} \cdot \nabla \vec{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u}$$

These mechanisms become __directionally biased__ in a thin shear layer:



CONVECTION (unchanged) PRESSURE (transverse component suppressed) DIFFUSION (transverse component accentuated)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \simeq -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 \simeq -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

1) Transverse velocity v is governed primarily by kinematic (continuity) requirements: $\partial v/\partial y = -\partial u/\partial x$, __not__ by dynamic ($y$-momentum) requirements. The y-momentum equation __decouples__ and is neglected.

2) Streamwise diffusion is negligible compared to transverse diffusion.

In real situations, assumption 1) is weaker than 2).