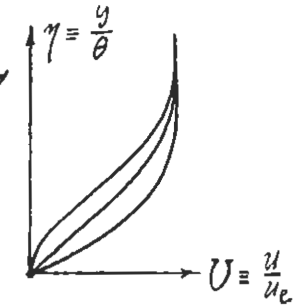
THWAITES' METHOD (For calculating laminar, incompressible, BL development)

Task: Given $u_e(x)$, ν, determine $\theta(x)$, $\delta^*(x)$, $C_f(x)$ ... Integrate $\frac{d\theta}{dx} = \frac{C_f}{2} - (H+2)\frac{\theta}{u_e}\frac{du_e}{dx}$

Problem ... need to relate C_f, H on r.h.s. to θ, u_e to allow integration.

Approach: Use assumed profile family e.g. Falkner-Skan
(note different η definition !)



Profiles are characterized by:

a) $\left.\frac{d^2U}{d\eta^2}\right|_{\eta=0}$ profile curvature at wall (related to $\frac{du_e}{dx}$)

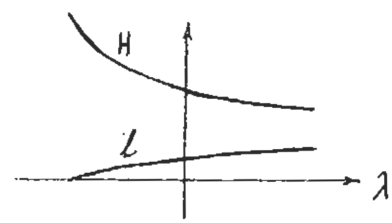
b) $\left.\frac{dU}{d\eta}\right|_{\eta=0}$ profile slope at wall (related to C_f)

c) $\int(1-U)d\eta \Big/ \int(1-U)U\,d\eta$ shape parameter $(= \delta^*/\theta)$

a) $\left.\frac{d^2U}{d\eta^2}\right|_{\eta=0} = \frac{\theta^2}{u_e}\frac{\partial^2 u}{\partial y^2} = \frac{\theta^2}{\nu u_e}\nu\frac{\partial^2 u}{\partial y^2} = -\frac{\theta^2}{\nu}\frac{du_e}{dx} \equiv -\lambda$ Thwaites parameter

b) $\left.\frac{dU}{d\eta}\right|_{\eta=0} = \frac{\theta}{u_e}\frac{\partial u}{\partial y} = \frac{u_e\theta}{\nu}\frac{\rho\nu\,\partial u/\partial y}{\rho u_e^2} = \frac{u_e\theta}{\nu}\frac{C_f}{2} \equiv \ell$

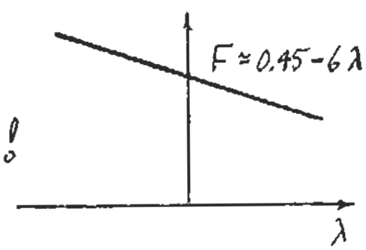
c) $\int(1-U)d\eta \Big/ \int(1-U)U\,d\eta = \int(1-U)dy \Big/ \int(1-U)U\,dy = \frac{\delta^*}{\theta} \equiv H$



Thwaites' assumption: ℓ, H only depend on λ. i.e. $\ell = \ell(\lambda)$, $H = H(\lambda)$

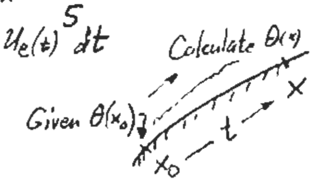
Integral momentum equation: $\frac{d\theta}{dx} = \frac{\nu}{u_e\theta}\ell - (H+2)\frac{\nu}{u_e\theta}\lambda$

or $\frac{u_e}{\nu}\frac{d(\theta^2)}{dx} = 2\left[\ell - (H+2)\lambda\right] \equiv F$ by luck, $F(\lambda)$ is nearly linear !



$\left\{\frac{u_e}{\nu}\frac{d\theta^2}{dx} \simeq 0.45 - 6\frac{\theta^2}{\nu}\frac{du_e}{dx}\right\} u_e^5$ ← integrating factor

$\frac{1}{\nu}\frac{d}{dx}\left(u_e^6\theta^2\right) = 0.45\,u_e^5 \longrightarrow u_e(x)^6\,\theta(x)^2 - u_e(x_0)^6\,\theta(x_0)^2 = 0.45\,\nu\int_{x_0}^{x} u_e(t)^5\,dt$

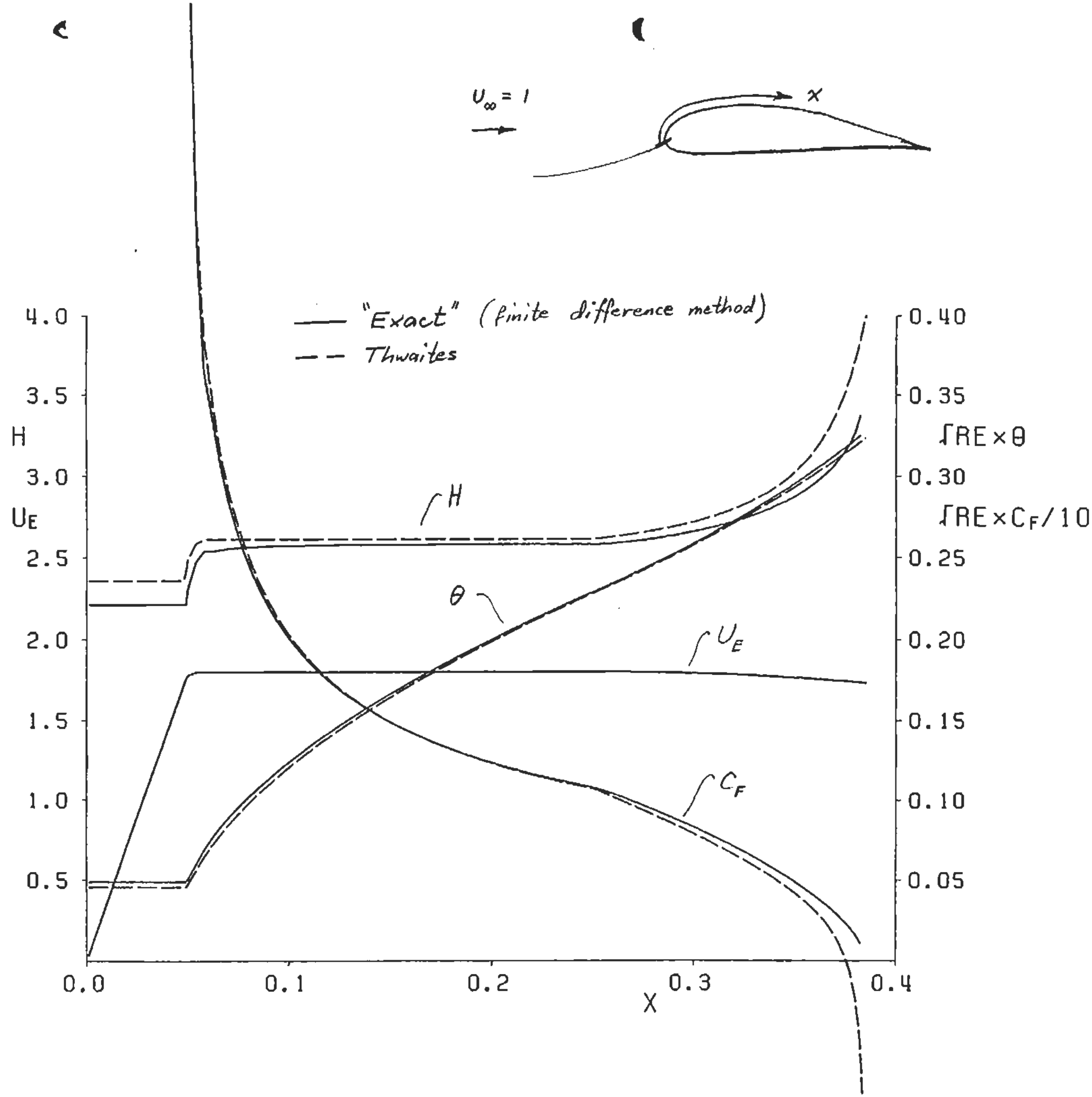


$$\boxed{\theta(x)^2 = \frac{1}{u_e(x)^6}\left\{\underbrace{u_e(x_0)^6\,\theta(x_0)^2}_{=0 \text{ at leading edge}} + 0.45\,\nu\int_{x_0}^{x} u_e(t)^5\,dt\right\}}$$

Once $\theta(x)$ is known, then $\lambda(x) = \frac{\theta(x)^2}{\nu}\frac{du_e(x)}{dx}$, $\ell(x) = \ell(\lambda(x))$, $H(x) = H(\lambda(x))$

Finally, $\delta^*(x) = H(x)\cdot\theta(x)$, $C_f(x) = 2\frac{\nu}{u_e(x)\theta(x)}\ell(x)$

use curve-fit functions



$U_\infty = 1$
x
"Exact" (finite difference method)
Thwaites
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
H
U_E
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
$\sqrt{RE} \times \theta$
$\sqrt{RE} \times C_F/10$
H
θ
U_E
C_F
0.0
0.1
0.2
0.3
0.4
X