$$\|e_n\| \leq \sum_{j=0}^{n-1} C(\Delta t)^{k+1} = nC(\Delta t)^{k+1}$$

$$\leq C\, t_n\, (\Delta t)^{k}, \quad k>0$$

$$\Longrightarrow \underline{\|e_n\| \to 0 \quad \text{as } \Delta t \to 0} \quad \text{qed}$$

LAX EQUIVALENCE THM

Stability properties of Trapezoidal rule

$$A\dot{y} + By = 0 \quad , \quad y(0) = y_0$$

$$F(\Delta t) = [A + \alpha \Delta t B]^{-1} [A - (1-\alpha)\Delta t B]$$

$$A = A^T, \; A > 0 \qquad B = B^T, \; B > 0$$

$$\|y\|^2 = y^T A y \qquad \dim y = N$$

Scalar case $\quad \dot{y} + \lambda y = 0 \quad , \; y(0) = y_0$

$$y_{n+1} = \mu(\Delta t)\, y_n \; , \; \mu(\Delta t) = \frac{1 - (1-\alpha)\lambda \Delta t}{1 + \alpha \lambda \Delta t}$$

unconditional stability for $\alpha \geq 1/2$

conditional stability for $\alpha < \frac{1}{2}$, $\Delta t_c = \frac{\lambda^{-1}}{1-2\alpha}$

Reduce the gral case to scalar cases by a spectral (modal) decomposition.

EVP: $(B - \lambda A)\varphi = 0$; $\|\varphi\| = 1$

$\Longrightarrow$ N solns (φ_r, λ_r), $r = 1, \ldots, N$

Props: 1) $\lambda_r > 0$ (from positive definiteness)

2) $\varphi_r A \varphi_s = \delta_{rs}$

3) $\{\varphi_r = 1, \ldots N\}$ forms a basis of $\mathbb{R}^N$

$\Longrightarrow \forall y \in \mathbb{R}^N \quad y = \sum_{r=1}^{N} y^r \varphi_r$ uniquely

y^r: modal coordinates.

$$y^T A \varphi_s = \left(\sum_{r=1}^{N} y^r \varphi_r\right)^T A \varphi_s = y^s$$

Eigenprojections: $\underset{\in \mathbb{R}^{N \times N}}{P_r} = \varphi_r \otimes \varphi_r = \varphi_r \varphi_r^T$

Props.: 1) $P_r^T = P_r$

2) $P_r^2 = (\varphi_r \varphi_r^T)(\varphi_r \varphi_r^T) = \varphi_r \varphi_r^T = P_r$

3) $\sum_{r=1}^{N} P_r = I$

$$P_r y = \varphi_r \varphi_r^T \left(\sum_{s=1}^{N} y^s \varphi_s\right) = \varphi_r \sum_{s=1}^{N} y^s \delta_{rs}$$

$$= y^r \varphi_r \quad \text{(no sum)}$$

(modal components)

Eq. of evolution: $A\dot{y} + By = 0$

$$A\left(\sum_{s=1}^{N} \dot{y}^s \varphi_s\right) + B\left(\sum_{s=1}^{N} y^s \varphi_s\right) = 0 \qquad (\;)\,\varphi_r^T$$

$$\longrightarrow \quad \boxed{\dot{y}^r + \lambda_r y^r = 0 \quad ; \quad r = 1, \ldots, N}$$

equations of evolution decouple mode by mode.

$$\|y\|^2 = y^T A y = \left(\sum_{r=1}^{N} y^r \varphi_r\right)^T A \left(\sum_{s=1}^{N} y^s \varphi_s\right)$$

$$\sum_{r=1}^{N} \sum_{s=1}^{N} y^r y^s \underbrace{\phi_r^T A \phi_s}_{\delta_{rs}}$$

$$\underline{\|y\|^2 = \sum_{r=1}^{N} |y^r|^2}$$ energy norm decouples mode by mode.

Trapezoidal rule also decouples mode by mode

$$A \frac{y_{n+1} - y_n}{\Delta t} + B\left[(1-\alpha) y_n + \alpha y_{n+1}\right] = 0$$

Let $y_{n+1} = \sum_{r=1}^{N} y_{n+1}^r \phi_r$, $y_n = \sum_{r=1}^{N} y_n^r \phi_r$

Insert and multiply through by ϕ_s^T, orthogonality

$$\frac{y_{n+1}^s - y_n^s}{\Delta t} + \lambda_s\left[(1+\alpha) y_n^s + \alpha y_{n+1}^s\right] = 0$$

Commutative diagram:

$$\begin{tikzcd}
y_n \arrow[r, "F(\Delta t)"] \arrow[d, "\text{Spectral decomp.}"'] & y_{n+1} \\
y_n^s \arrow[r] & y_{n+1}^s \arrow[u, "\text{spectral synthesis}"']
\end{tikzcd}$$

$\Rightarrow$ algorithm never mixes energy between different modes

Summary IVP EVP

$$\left.\begin{aligned} A\dot{y} + By &= 0 \\ y(0) &= y_0 \end{aligned}\right\} \qquad \begin{aligned} (B - \lambda A)\varphi &= 0 \\ P_r &= \varphi_r \otimes \varphi_r \end{aligned}$$

Spectral decomposition: $y = \sum_{r=1}^{N} y^r \varphi_r$ (SD)

(i) $\rightarrow A\dot{y} + By = 0 \xrightarrow{SD} \dot{y}^r + \lambda_r y^r = 0$

$$\boxed{e^{-tA^{-1}B} P_r = P_r e^{-tA^{-1}B}}$$

propagator commutes with eigenprojections!

$$e^{-tA^{-1}B} P_r y_0 = P_r e^{-tA^{-1}B} y_0 = P_r y(t) \quad !!$$

(ii) $A \dfrac{y_{n+1} - y_n}{\Delta t} + B y_{n+\alpha} = 0 \xrightarrow{SD} \dfrac{y_{n}^r - y_{n}^r}{\Delta t} + \lambda_r y_{n+\alpha}^r = 0$

i.e. $F(\Delta t)\, P_r = P_r\, F(\Delta t)$

amplification matrix commutes with P_r

$$F(\Delta t)\, P_r\, y_n = P_r\, F(\Delta t)\, y_n$$

(iii) <u>Energy norm</u>: $\|y\|^2 = \sum_{r=1}^{N} |y^r|^2$

stability condition $\|y_{n+1}\| < \|y_n\| \quad \forall y_n$

$$\Longrightarrow \sum_{r=1}^{N} |y_{n+1}^r|^2 < \sum_{r=1}^{N} |y_n^r|^2 \qquad \forall y_n$$

$$\Longrightarrow \quad |y_{n+1}^r| < |y_n^r| \quad , r = 1, \ldots, N$$

Since $y_{n+1}^r = \mu_r(\Delta t)\, y_n^r \Longrightarrow |\mu_r(\Delta t)| < 1 \;, r = 1, \ldots, N$

$\alpha \geqslant \frac{1}{2} \Longrightarrow$ unconditional stability

$\alpha < 1/2 \Longrightarrow$ conditional stability $\Longrightarrow$

$$\Delta t \leq \frac{2}{(1-2\alpha)\lambda_r} \qquad r = 1, \dots, N$$

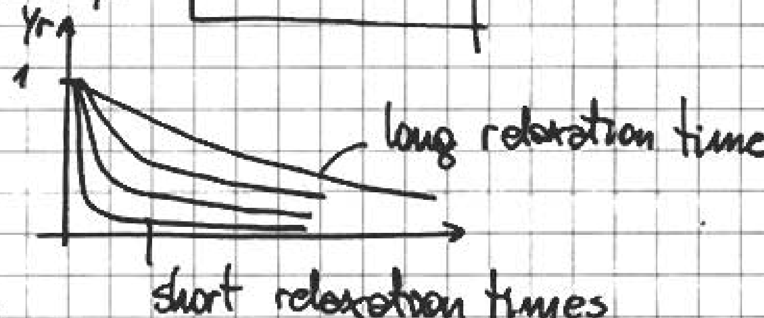
$$\boxed{\Delta t \leq \Delta t_c = \frac{2}{(1-2\alpha)\lambda_{max}}}$$

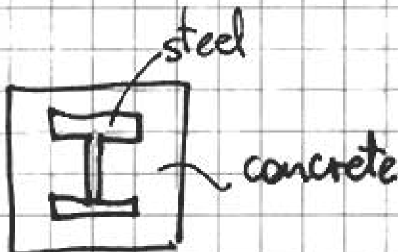
Stability governed by λ_{max}.

Choice of Δt:

i) Conditional stability $\boxed{\Delta t \ll \Delta t_c}$

ii) Stiff systems





Say one knows a priori that $y_0^r \sim 0$ $\quad r \geq m$

$\Longrightarrow$ implicit methods

choose Δt to resolve the smallest relevant relax. time

$$\boxed{\Delta t < \frac{2}{\lambda_m}}$$

Stability of Newmark's Algorithm

IVP $\left.\begin{array}{l} M\ddot{x} + C\dot{x} + Kx = 0 \\ x(0) = x_0 \,,\ \dot{x}(0) = v_0 \end{array}\right\}$

EVP: $(K - \omega^2 M)\, q = 0$ $\quad \omega \equiv$ eigenfrequencies.

Spectral dec $\quad x = \sum_{r=1}^{N} x^r q_r$

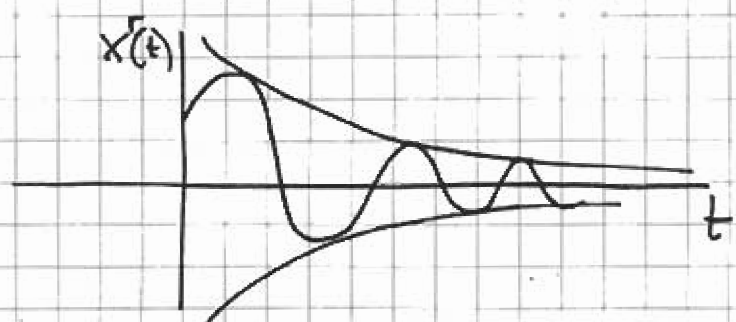
Assume C has same eigenvectors as K.

$$\ddot{x}^r + 2\xi_r \omega_r \dot{x}^r + \omega_r^2 x_r = 0$$

$\rightarrow$ eqns of evolution decouple mode by mode.

$$x^r(t) = e^{-\xi_r \omega_r t}\left(C_1^r e^{i\omega_D^r t} + C_2^r e^{-i\omega_D^r t}\right)$$

$$\omega_D^r = \omega^r \sqrt{1 - \xi^r}$$



$T = \frac{2\pi}{\omega_D}$ = period

(Hughes, 1983)

$$\left.\begin{aligned} x_{n+1} &= x_n + \Delta t\, v_n + \left[\left(\tfrac{1}{2}-\beta\right) a_n + \beta\, a_{n+1}\right]\Delta t^2 \\ v_{n+1} &= v_n + \Delta t\left[(1-\gamma)\, a_n + \gamma\, a_{n+1}\right] \\ M\, a_{n+1} &+ C\, v_{n+1} + K\, x_{n+1} = 0 \end{aligned}\right\} \text{New'. Alg.}$$

$$\underline{\underline{SD}} \rightarrow \left\{\begin{aligned} x^r_{n+1} &= x^r_n + \Delta t\, v^r_n + \left[\left(\tfrac{1}{2}-\beta\right) a^r_n + \beta\, a^r_{n+1}\right]\Delta t^2 \\ v^r_{n+1} &= v^r_n + \Delta t\left[(1-\gamma)\, a^r_n + \gamma\, a^r_{n+1}\right] \\ \cancel{M a_{n+1} + C v_{n+1}}\ & a^r_{n+1} + 2\xi_r\, \omega_r\, v^r_{n+1} + \omega_r^2\, x^r_{n+1} = 0 \end{aligned}\right.$$

$r = 1, \dots, N$

→ Newmark's eqs. decouple mode by mode!

Amplification matrix (drop "r")

$$\underbrace{\begin{pmatrix} 1+\beta\Omega^2 & 2\beta\xi\Omega^2 \\ \gamma\Omega & 1+2\gamma\xi\Omega \end{pmatrix}^{-1}}_{P^{-1}} \underbrace{\begin{pmatrix} 1-\left(\tfrac{1}{2}-\beta\right)\Omega^2 & \Omega - 2\left(\tfrac{1}{2}-\beta\right)\xi\Omega^2 \\ -(1-\gamma)\Omega & 1-2(1-\gamma)\xi\Omega \end{pmatrix}}_{Q}$$

$F(\Delta t)$ $\qquad \Omega = \omega \Delta t$

$$\begin{cases} \|F(\Delta t)\| < 1 \Rightarrow \text{eigenvalues } |\mu_{1,2}| < 1 \\ \mu_{1,2} = A_1 \pm \sqrt{A_1^2 - A_2} \\ A_1 = 1 - \left[\frac{1}{2}\left(\gamma + \frac{1}{2}\right)\Omega^2 + \xi\Omega\right]/D \\ A_2 = 1 - \left[\left(\gamma - \frac{1}{2}\right)\Omega^2 + 2\xi\Omega\right]/D \\ D = 1 + 2\gamma\xi\Omega + \beta\Omega^2 \end{cases}$$

<u>Classification of regimes:</u>

i) $\boxed{A_1^2 < A_2}$ $\Rightarrow$ $\mu_{1,2}$ are complex conjugate

Express $\mu_{1,2} = e^{-\bar{\xi}\bar{\omega}\Delta t} \; e^{\pm i \bar{\omega}_D \Delta t}$

$$\Longrightarrow y_n = e^{-\bar{\xi}\bar{\omega} t_n}\left(C_1 e^{i\bar{\omega}_D t_n} + C_2 e^{-i\bar{\omega}_D t_n}\right)$$

which has the same form as $y(t_n)$ but errors in $\bar{\xi}, \bar{\omega}$.

$$\bar{\xi} = -\log A_2 / 2\bar{\Omega}$$

Say $\xi = 0$ $\quad \underline{\bar{\xi}} = \frac{1}{2}\left(\gamma - \frac{1}{2}\right)\Omega + O(\Omega^3)$

$\boxed{\gamma > 1/2}$ <u>numerical dissipation</u>

damps more the higher modes.
(may be beneficial)

$\boxed{\bar{T} = \frac{2\pi}{\bar{\omega}_D}} \neq T = \frac{2\pi}{\omega_D}$, Typically $\bar{T} > T$
<u>(period elongation)</u>

$\boxed{\gamma \geq \frac{1}{2} \quad \beta \geq \frac{\gamma}{2}} \Longrightarrow$ unconditional stability

$\beta \geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \Longrightarrow$ oscillatory solution

Otherwise

$$\boxed{\Delta t_c = \frac{\xi\left(\gamma - \frac{1}{2}\right) + \left[\frac{\gamma}{2} - \beta + \xi^2\left(\gamma - \frac{1}{2}\right)^2\right]^{1/2}}{\left(\frac{\gamma}{2} - \beta\right)} \frac{1}{\omega_{max}}}$$

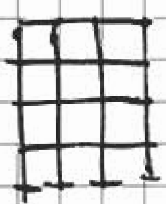
<u>choice of time step</u>

i) Explicit dynamics $\quad \beta = 0 \quad \gamma = 1/2$

$\boxed{\Delta t \sim \Delta t_c}$

ii) <u>Stiff systems</u> $x^r \sim 0$ for $r \geq m$

(structural dynamics)

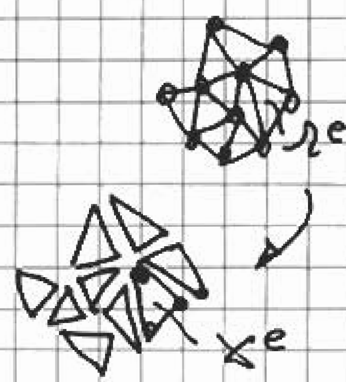


$\longrightarrow$ Implicit dynamics

$$\Delta t \sim \frac{T_m}{20}$$

<u>Iron's bounding principle</u>

$$\omega^2_{max} = \max_{x \in \mathbb{R}^N} \frac{x^T K x}{x^T M x}$$



Consider extended

$$\bar{x} = \left\{ x^e \right\}_{e=1}^{E} \quad \text{incompatible}$$

$$\bar{K} = \text{diag}\left\{ K^e \right\}, \quad \bar{M} = \text{diag}\left\{ M^e \right\}$$

If $\bar{x}$ compatible $\Longrightarrow$ $\dfrac{\bar{x}^T \bar{K} \bar{x}}{\bar{x}^T \bar{M} \bar{x}} = \dfrac{x^T K x}{x M x}$

$$\bar{\omega}^2_{max} = \max_{\bar{x}} \frac{\bar{x}^T \bar{K} \bar{x}}{\bar{x}^T \bar{M} \bar{x}} \geqslant \omega^2_{max}$$

why? because $\bar{x}$ are a subset of x.

$$\Rightarrow \bar{\omega}^2_{max} = \max_{\bar{x}} \frac{\sum_e (x^e)^T K^e x^e}{\sum_e (x^e)^T M^e x^e}$$

$$\leqslant \max_e \left[\max_{x^e} \frac{(x^e)^T K^e x^e}{(x^e)^T M^e x^e} \right] = \max_e \left(\omega^{e\,2}_{max} \right)$$

$$\boxed{\bar{\omega}^2_{max} = \max_e \left(\omega^{e\,2}_{max} \right)}$$

max. eigenfrequency of element e.