Trapezoidal rule - Heat conduction

$$C\dot{\theta} + f^{int}(\theta) = f^{ext}(t) \quad , \quad \theta(0) = \theta_0$$

$$\boxed{C\,\frac{\theta_{n+1}-\theta_n}{\Delta t} + \left[(1-\alpha)\,f^{int}(\theta_n) + \alpha\, f^{int}(\theta_{n+1})\right] = \left[(1-\alpha)\,f^{ext}_{n} + \alpha\, f^{ext}_{n+1}\right]}$$

Solve by Newton-Raphson method.

Rewrite algorithm in a manner similar to Newmark

$$\frac{\theta_{n+1}-\theta_n}{\Delta t} = C^{-1}\left[(1-\alpha)\left(f^{ext}-f^{int}\right) + \alpha\left(f^{ext}_{n+1}-f^{int}_{n+1}\right)\right]$$

Observe $C^{-1}\left(f^{ext}-f^{int}\right) = \dot{\theta}$

$$\Rightarrow \frac{\theta_{n+1}-\theta_n}{\Delta t} = (1-\alpha)\,\dot{\theta}_n + \alpha\,\dot{\theta}_{n+1}$$

$$\boxed{\left.\begin{aligned} \theta_{n+1} &= \theta_n + \Delta t\left[(1-\alpha)\,\dot{\theta}_n + \alpha\,\dot{\theta}\right] \\ C\,\dot{\theta}_{n+1} + f^{int}(\theta_{n+1}) &= f^{ext}_{n+1} \end{aligned}\right| \Rightarrow \begin{matrix}\theta_{n+1} \\ \dot{\theta}_{n+1}\end{matrix}}$$

① Trapezoidal predictor

$$\theta_{n+1}^{(0)} = \theta_n + (1-\alpha)\Delta t\,\dot{\theta}_n$$

② Equivalent static problem: $\left(\theta_{n+1}^{(k)}, \dot{\theta}_{n+1}^{(k)}\right) \rightarrow \left(\theta_{n+1}^{(k+1)}, \dot{\theta}_{n+1}^{(k+1)}\right)$

Linearize both equations about $\theta_{n+1}^{(k)}$

$$\theta_{n+1}^{(k+1)} = \theta_{n+1}^{(k)} + \Delta\theta$$

from first equation: $\Delta\theta = \alpha\,\Delta t\,\Delta\dot{\theta}$, $\Delta\dot{\theta} = \dfrac{\Delta\theta}{\alpha\Delta t}$

second:

$$C\left(\dot{\theta}_{n+1}^{(k)} + \Delta\dot{\theta}\right) + \underbrace{f^{int}\left(\theta_{n+1}^{(k)} + \Delta\theta\right)}_{f^{int}\left(\theta_{n+1}^{(k)}\right) + \left.\frac{\partial f}{\partial \theta}\right|_{\theta_{n+1}^{(k)}} \Delta\theta} = f_{n+1}^{ext}$$

$$f^{int}\left(\theta_{n+1}^{(k)}\right) + \underbrace{\left.\frac{\partial f}{\partial \theta}\right|_{\theta_{n+1}^{(k)}}}_{K\left(\theta_{n+1}^{(k)}\right)} \Delta\theta = f_{n+1}^{ext}$$

$$\underbrace{\left(\frac{1}{\alpha\Delta t}C + K_{n+1}^{(k)}\right)}_{K^{eff}} \Delta\theta = \underbrace{f_{n+1}^{ext} - f^{int}\left(\theta_{n+1}^{(k)}\right) - C\,\dot{\theta}_{n+1}^{(k)}}_{\Gamma_{n+1}^{(k)}}$$

$$\boxed{K^{eff}\,\Delta\theta = r_{n+1}^{(k)}}$$

③ Trapezoidal correctors

$$\theta_{n+1}^{(k+1)} = \theta_{n+1}^{(k)} + \Delta\theta$$

$$\dot{\theta}_{n+1}^{(k+1)} = \dot{\theta}_{n+1}^{(k)} + \frac{\Delta\theta}{\alpha\,\Delta t}$$

④ Convergence check

$$\| r_{n+1}^{(k+1)} \| \leq TOL\ \| r_{n+1}^{(0)} \| \ ?\ ⑤ : ②,\ k \leftarrow k+1$$

⑤ $n \leftarrow n+1$ until $t_{n+1} = t_{max}$

The trapezoidal rule is implicit. Special cases:

- $\alpha = 0$: explicit, Forward Euler
- $\alpha = 1$: implicit, Backward Euler

- $\alpha = 0$: $\theta_{n+1} = \theta_n + \Delta t\,\dot{\theta}_n$

$$\dot{\theta}_{n+1} = C^{-1}\left(f^{ext}_{n+1} - f^{int}(\theta_{n+1})\right)$$

If C is diagonal there is no equation solving.

Initial rates: Necessary for algorithm start.

Dynamics: $a_0 = M^{-1}\left(f_0^{ext} - f^{int}(x_0, v_0)\right)$

Heat conduction: $\dot{\theta}_0 = C^{-1}\left(f_0^{ext} - f^{int}(\theta_0)\right)$

Other algorithms: Multi-step methods

$$\boxed{C\,\frac{\theta_{n+1}-\theta_n}{\Delta t} = \sum_{j=-p}^{1} \alpha_j \left(f^{ext}_{n+j} - f^{int}_{n+j}\right)}$$

Trapezoidal rule: $\alpha_1 = \alpha$, $\alpha_0 = (1-\alpha)$, $p=0$