APPENDIX G

ANALYTICAL JUSTIFICATION FOR THE FILTER HYPOTHESIS[1]

Presentation of the following development is motivated primarily by two considerations. First, it gives a mathematical setting for the *filter hypothesis.* This condition is arrived at directly from the system differential equation, as opposed to the physical or intuitive presentation in Chap. 3. Second, and quite importantly, this development points out that what one actually determines with the DF method is an *approximation to the first harmonic* of the true nonlinear oscillation, *not* the oscillation waveform itself.

Consider any autonomous nonlinear system which can be reduced to single-loop block-diagram form, including a single nonlinearity with input x and output $y(x,\dot{x})$, and a linear part $L(s)$, where

$$L(s) = \frac{P(s)}{Q(s)} \tag{G-1}$$

$P(s)$ and $Q(s)$ are polynomials of arbitrary degree in s, the degree of $P(s)$ being lower than the degree of $Q(s)$. The equation defining the modes of behavior of the closed-loop system is

$$Q(s)x + P(s)y(x,\dot{x}) = 0 \tag{G-2}$$

[1] This derivation closely follows that of Popov and Pal'tov (Chap. 6, Ref. 21, pp. 121–129, in the English translation.)

where $s = d/dt$. According to the DF point of view, the limit cycle behavior of this system is studied in terms of the existence of the solution

$$x = A \sin \omega t \tag{G-3}$$

to the linearized equation [cf. Eq. (2.2-30)]

$$\left\{Q(s) + P(s)\left[n_p(A,\omega) + \frac{n_q(A,\omega)}{\omega}s\right]\right\}x = 0 \tag{G-4}$$

We now attempt to find the conditions which $P(s)$ and $Q(s)$ must satisfy in order that Eq. (G-2) will be sufficiently close to Eq. (G-4) in the presence of a strong nonlinearity, $y(x,\dot{x})$, when we seek a periodic solution $x(t)$ close to sinusoidal.

Let us therefore seek a solution to the given nonlinear differential equation (G-2) in the form

$$x(t) = x_1(t) + \epsilon x_r(t) \tag{G-5}$$

where

$$x_1(t) = A_1 \sin \omega_1 t \tag{G-6}$$

is the true first harmonic of the periodic solution, $x_r(t)$ is an arbitrary bounded time function, and ϵ is a small parameter, viz.,

$$\epsilon x_r(t) = \epsilon \sum_{k=2}^{\infty} A_k \sin(k\omega_1 t + \theta_k) \tag{G-7}$$

Substituting for x from Eq. (G-5) into Eq. (G-2), we obtain

$$Q(s)(x_1 + \epsilon x_r) + P(s)y(x_1 + \epsilon x_r,\ \dot{x}_1 + \epsilon \dot{x}_r) = 0 \tag{G-8}$$

Next, the nonlinearity output is expanded in the Taylor series (truncated at first order in ϵ):

$$y(x_1 + \epsilon x_r,\ \dot{x}_1 + \epsilon\dot{x}_r) \simeq y(x_1,\dot{x}_1) + \epsilon\left[\frac{\partial y(x_1,\dot{x}_1)}{\partial x}x_r + \frac{\partial y(x_1,\dot{x}_1)}{\partial \dot{x}}\dot{x}_r\right] \tag{G-9}$$

This expression states, in essence, that the periodic nonlinearity output in response to $x(t)$ is very close (i.e., to an order ϵ) to the response produced by the first harmonic, $x_1(t)$, alone. This will be true even for the discontinuous nonlinearities often encountered.

The first term in the right-hand member of Eq. (G-9) is the nonlinearity output when forced by a sinusoid of amplitude A_1 and frequency ω_1. In DF notation, we have

$$y(x_1,\dot{x}_1) = F_0 + A_1\left[n_p(A_1,\omega_1) + \frac{n_q(A_1,\omega_1)}{\omega_1}s\right]\sin \omega_1 t + \sum_{k=2}^{\infty} F_k \sin(k\omega_1 t + \varphi_k) \tag{G-10}$$

where, as $k \to \infty$, we require $F_k \to 0$. The second term in the right-hand member of Eq. (G-9) is represented by its Fourier series, namely,

$$\epsilon\left[\frac{\partial y(x_1,\dot{x}_1)}{\partial x} x_r + \frac{\partial y(x_1,\dot{x}_1)}{\partial \dot{x}} \dot{x}_r\right] = \epsilon \sum_{k=0}^{\infty} G_k \sin (k\omega_1 t + \lambda_k) \quad \text{(G-11)}$$

Substituting the expressions of Eqs. (G-6), (G-7), and (G-9) to (G-11) into (G-8) gives

$$Q(s)A_1 \sin \omega_1 t + \epsilon Q(s) \sum_{k=2}^{\infty} A_k \sin (k\omega_1 t + \theta_k) + P(s)F_0$$
$$+ P(s)A_1\left[n_p(A_1,\omega_1) + \frac{n_q(A_1,\omega_1)}{\omega_1} s\right] \sin \omega_1 t + P(s) \sum_{k=2}^{\infty} F_k \sin (k\omega_1 t + \varphi_k)$$
$$+ \epsilon P(s) \sum_{k=0}^{\infty} G_k \sin (k\omega_1 t + \lambda_k) = 0 \quad \text{(G-12)}$$

In order to satisfy equality with zero, the coefficient of each harmonic in the left-hand member of Eq. (G-12) must be individually set to zero.

ZERO HARMONIC

From Eq. (G-12) we require

$$F_0 + \epsilon G_0 \sin \lambda_0 = 0 \quad \text{(G-13)}$$

or to an order ϵ,

$$F_0 = 0 \quad \text{(G-14)}$$

Thus we have the first requirement imposed upon $y(x,\dot{x})$: the absence of a constant component. This is consistent with the content of Chap. 3 (although in Chap. 6 this restriction is removed).

FIRST HARMONIC

From Eq. (G-12) we require, upon division by $Q(s)$,

$$A_1 \sin \omega_1 t + L(s)A_1\left[n_p(A_1,\omega_1) + \frac{n_q(A_1,\omega_1)}{\omega_1} s\right] \sin \omega_1 t$$
$$+ \epsilon L(s)G_1 \sin (\omega_1 t + \lambda_1) = 0 \quad \text{(G-15)}$$

or to an order ϵ,

$$A \sin \omega t + L(s)A\left[n_p(A,\omega) + \frac{n_q(A,\omega)}{\omega} s\right] \sin \omega t = 0 \quad \text{(G-16)}$$

This may be rewritten as

$$1 + L(j\omega)[n_p(A,\omega) + jn_q(A,\omega)] = 0 \quad \text{(G-17)}$$

which is the familiar DF statement defining a system limit cycle. Hence, to an order ϵ, the DF method determines the *first harmonic* of the periodic solution of the nonlinear system [Eq. (G-2)], which is close to a sinusoidal solution (if this periodic solution exists).

HIGHER HARMONICS

For $k = 2, 3, \ldots$, it follows from Eq. (G-12) that the kth harmonic terms must satisfy

$$\epsilon A_k \sin(k\omega_1 t + \theta_k) + L(jk\omega_1)F_k \sin(k\omega_1 t + \varphi_k) + \epsilon L(jk\omega_1)G_k \sin(k\omega_1 t + \lambda_k) = 0 \quad \text{(G-18)}$$

In order to observe this equality, that is, in order that all the terms $\epsilon A_k \sin(k\omega_1 t + \theta_k)$ may in reality be small, we require small order of magnitude of the terms $|L(jk\omega_1)|\, F_k$. Each term must be at least of order ϵ; that is, in comparison with A_1, each term must be at least of the same order of magnitude as the quantity ϵx_r in comparison with x_1. Thus, employing Eq. (G-17), we establish as a requirement the inequality

$$|L(jk\omega)|\, F_k \ll |L(j\omega)|\, A\sqrt{n_p^2(A,\omega) + n_q^2(A,\omega)} \qquad \text{(G-19)}$$

Since we are considering strong nonlinearities, it does not follow that the quantities F_k may be considered small in comparison with $A\sqrt{n_p^2 + n_q^2}$, particularly for lower values of k. Therefore we must require that

$$|L(jk\omega)| \ll |L(j\omega)| \qquad \text{for } k = 2, 3, \ldots \qquad \text{(G-20)}$$

and since the degree of $Q(s)$ exceeds the degree of $P(s)$,

$$|L(jk\omega)| \to 0 \qquad \text{as } k \to \infty \qquad \text{(G-21)}$$

Equations (G-20) and (G-21) constitute the so-called "filter hypothesis."