Topic #6

16.30/31 Feedback Control Systems

State-Space Systems

- What are state-space models?
- Why should we use them?
- **How are they related to the transfer functions used in classical control design and how do we develop a state-space model?**
- What are the basic properties of a state-space model, and how do we analyze these?

TF's to State-Space Models

- The goal is to develop a state-space model given a transfer function for a system $G(s)$.
  - There are many, many ways to do this.

- But there are three primary cases to consider:
  1. Simple numerator (strictly proper)
  $$\frac{y}{u} = G(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$
  2. Numerator order less than denominator order (strictly proper)
  $$\frac{y}{u} = G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{N(s)}{D(s)}$$
  3. Numerator equal to denominator order (proper)
  $$\frac{y}{u} = G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

- These 3 cover all cases of interest

- Consider **case 1** (specific example of third order, but the extension to n^{th} follows easily)

$$\frac{y}{u} = G(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

can be rewritten as the differential equation

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = u$$

choose the output y and its derivatives as the state vector

$$\mathbf{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}$$

then the state equations are

$$\dot{\mathbf{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} + [0]u$$

- This is typically called the *controller form* for reasons that will become obvious later on.
  - There are four classic (called *canonical*) forms – observer, controller, controllability, and observability. They are all useful in their own way.

- Consider **case 2**

$$\frac{y}{u} = G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{N(s)}{D(s)}$$

- Let

$$\frac{y}{u} = \frac{y}{v} \cdot \frac{v}{u}$$

where $y/v = N(s)$ and $v/u = 1/D(s)$

- Then representation of $v/u = 1/D(s)$ is the same as **case 1**

$$\dddot{v} + a_1 \ddot{v} + a_2 \dot{v} + a_3 v = u$$

use the state vector

$$\mathbf{x} = \begin{bmatrix} \ddot{v} \\ \dot{v} \\ v \end{bmatrix}$$

to get

$$\dot{\mathbf{x}} = A_2 \mathbf{x} + B_2 u$$

where

$$A_2 = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Then consider $y/v = N(s)$, which implies that

$$\begin{aligned} y &= b_1 \ddot{v} + b_2 \dot{v} + b_3 v \\ &= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \ddot{v} \\ \dot{v} \\ v \end{bmatrix} \\ &= C_2 \mathbf{x} + [0] u \end{aligned}$$

- Consider **case 3** with

$$\frac{y}{u} = G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + a_1 s^2 + a_2 s + a_3} + D$$

$$= G_1(s) + D$$

where

$$\begin{array}{lcccc} D(& s^3 & +a_1 s^2 & +a_2 s & +a_3\ ) \\ +(& & +\beta_1 s^2 & +\beta_2 s & +\beta_3\ ) \\ \hline\hline = & b_0 s^3 & +b_1 s^2 & +b_2 s & +b_3 \end{array}$$

so that, given the b_i, we can easily find the β_i

$$\begin{aligned} D &= b_0 \\ \beta_1 &= b_1 - D a_1 \\ &\vdots \end{aligned}$$

- Given the β_i, can find $G_1(s)$
  - Can make state-space model for $G_1(s)$ as in **case 2**

- Then we just add the "feed-through" term Du to the output equation from the model for $G_1(s)$

- Will see that there is a lot of freedom in making a state-space model because we are free to pick the $\mathbf{x}$ as we want

Modal Form

- One particular useful canonical form is called the **Modal Form**
  - It is a diagonal representation of the state-space model.
- Assume for now that the transfer function has distinct real poles p_i (easily extends to case with complex poles, see 7–??)

$$
\begin{aligned}
G(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\
&= \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}
\end{aligned}
$$

- Now define collection of first order systems, each with state x_i

$$
\begin{aligned}
\frac{X_1}{U(s)} &= \frac{r_1}{s-p_1} \Rightarrow \dot{x}_1 = p_1 x_1 + r_1 u \\
\frac{X_2}{U(s)} &= \frac{r_2}{s-p_2} \Rightarrow \dot{x}_2 = p_2 x_2 + r_2 u \\
&\vdots \\
\frac{X_n}{U(s)} &= \frac{r_n}{s-p_n} \Rightarrow \dot{x}_n = p_n x_n + r_n u
\end{aligned}
$$

- Which can be written as

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \\
y(t) &= C\mathbf{x}(t) + Du(t)
\end{aligned}
$$

with

$$
A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \quad B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T
$$

- Good representation to use for numerical robustness reasons.
  - Avoids some of the large coefficients in the other 4 canonical forms.

State-Space Models to TF's

- Given the Linear Time-Invariant (LTI) state dynamics

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t)\\ y(t) &= C\mathbf{x}(t) + Du(t)\end{aligned}$$

what is the corresponding transfer function?

- Start by taking the Laplace Transform of these equations

$$\begin{aligned}\mathcal{L}\{\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t)\}\\ sX(s) - \mathbf{x}(0^-) &= AX(s) + BU(s)\end{aligned}$$

$$\begin{aligned}\mathcal{L}\{y(t) &= C\mathbf{x}(t) + Du(t)\}\\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

which gives

$$\begin{aligned}(sI - A)X(s) &= BU(s) + \mathbf{x}(0^-)\\ \Rightarrow X(s) &= (sI - A)^{-1}BU(s) + (sI - A)^{-1}\mathbf{x}(0^-)\end{aligned}$$

and

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s) + C(sI - A)^{-1}\mathbf{x}(0^-)$$

- By definition $G(s) = C(sI - A)^{-1}B + D$ is called the **Transfer Function** of the system.

- And $C(sI - A)^{-1}\mathbf{x}(0^-)$ is the initial condition response.
  - It is part of the response, but not part of the transfer function.

SS to TF

- In going from the state space model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t)\\ y(t) &= C\mathbf{x}(t) + Du(t)\end{aligned}$$

  to the transfer function $G(s) = C(sI - A)^{-1}B + D$ need to form inverse of matrix $(sI - A)$
  - A symbolic inverse – not very easy.

- For simple cases, we can use the following:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1} = \frac{1}{a_1a_4 - a_2a_3}\begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$$

  For larger problems, we can also use *Cramer's Rule*

- Turns out that an equivalent method is to form:[1]

$$G(s) = C(sI - A)^{-1}B + D = \frac{\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}}{\det(sI - A)}$$

  - Reason for this will become more apparent later (see 8–??) when we talk about how to compute the "zeros" of a state-space model (which are the roots of the numerator)

- **Key point:** System characteristic equation given by

$$\phi(s) = \det(sI - A) = 0$$

  - It is the roots of $\phi(s) = 0$ that determine the poles of the system. Will show that these determine the time response of the system.

[1] see here

- Example from Case 2, page 6–4

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$$

then

$$\begin{aligned} G(s) &= \frac{1}{\det(sI - A)} \cdot \det \left[\begin{array}{ccc|c} s + a_1 & a_2 & a_3 & -1 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ \hline b_1 & b_2 & b_3 & 0 \end{array} \right] \\ &= \frac{b_3 + b_2 s + b_1 s^2}{\det(sI - A)} \end{aligned}$$

and $\det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3$

- Which is obviously the same as before.

State-Space Transformations

- State space representations are not unique because we have a lot of freedom in choosing the state vector.
  - Selection of the state is quite arbitrary, and not that important.
- In fact, given one model, we can *transform* it to another model that is **equivalent** in terms of its input-output properties.
- To see this, define Model 1 of $G(s)$ as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t)\end{aligned}$$

- Now introduce the new state vector $\mathbf{z}$ related to the first state $\mathbf{x}$ through the transformation $\mathbf{x} = T\mathbf{z}$
  - T is an invertible (similarity) transform matrix

$$\begin{aligned}\dot{\mathbf{z}} = T^{-1}\dot{\mathbf{x}} &= T^{-1}(A\mathbf{x} + B\mathbf{u})\\ &= T^{-1}(AT\mathbf{z} + B\mathbf{u})\\ &= (T^{-1}AT)\mathbf{z} + T^{-1}B\mathbf{u} = \bar{A}\mathbf{z} + \bar{B}\mathbf{u}\end{aligned}$$

and

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u} = CT\mathbf{z} + D\mathbf{u} = \bar{C}\mathbf{z} + \bar{D}\mathbf{u}$$

- So the new model is

$$\begin{aligned}\dot{\mathbf{z}} &= \bar{A}\mathbf{z} + \bar{B}\mathbf{u}\\ \mathbf{y} &= \bar{C}\mathbf{z} + \bar{D}\mathbf{u}\end{aligned}$$

- Are these going to give the same transfer function? They must if these really are equivalent models.

- Consider the two transfer functions:

$$\begin{aligned} G_1(s) &= C(sI - A)^{-1}B + D \\ G_2(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \end{aligned}$$

Does $G_1(s) \equiv G_2(s)$?

$$\begin{aligned} G_1(s) &= C(sI - A)^{-1}B + D \\ &= C(TT^{-1})(sI - A)^{-1}(TT^{-1})B + D \\ &= (CT)\left[T^{-1}(sI - A)^{-1}T\right](T^{-1}B) + \bar{D} \\ &= (\bar{C})\left[T^{-1}(sI - A)T\right]^{-1}(\bar{B}) + \bar{D} \\ &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = G_2(s) \end{aligned}$$

- So the transfer function is not changed by putting the state-space model through a similarity transformation.

- Note that in the transfer function

$$G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

we have 6 parameters to choose

- But in the related state-space model, we have $A - 3 \times 3$, $B - 3 \times 1$, $C - 1 \times 3$ for a total of 15 parameters.

- Is there a contradiction here because we have more degrees of freedom in the state-space model?
  - No. In choosing a representation of the model, we are effectively choosing a T, which is also 3×3, and thus has the remaining 9 degrees of freedom in the state-space model.