Recitation 12

16.30/31: Estimation and Control of Aerospace Systems

November 29, 2010

Consider a space telescope, and let $\vec{d}$ be a unit vector aligned with the telescope's line of sight. It is desired to point the telescope towards a star, in direction $\vec{d_0}$. The dynamics of the spacecraft are described by the equations of motion

$$J\dot{\vec{\omega}} + \vec{\omega} \times J\vec{\omega} = \vec{\tau},$$

where $\vec{\omega}$ is the angular velocity of the spacecraft (in body axes), J is its inertia tensor, and $\vec{\tau}$ is the control torque (in body axes). In body axes, the star's direction is not fixed, but rotates as

$$\dot{\vec{d_0}} = -\vec{\omega} \times \vec{d_0}.$$

Consider the control law

$$\vec{\tau} = -k\vec{\omega} + \vec{d} \times \vec{d_0}.$$

1. Find the equilibrium points for the spacecraft, under the given control law.
2. Consider the following function:

$$V(\vec{\omega}, \vec{d_0}) = \frac{1}{2}\vec{\omega} \cdot J\vec{\omega} + \frac{1}{2}|\vec{d} - \vec{d_0}|^2.$$

   Is it a good candidate for a Lyapunov function? Study the stability of the equilibrium point(s).

   It may be interesting for you to simulate the system in Matlab, and look at the trajectories of the "star vector" d_0. Notice that the control law is the sum of a "derivative" term $k\vec{\omega}$ and a "proportional" term $\vec{d} \times \vec{d_0}$. How do trajectories change as you choose different values of k?

Solution: *First of all, let us write the dynamics of the system, under the given control law:*

$$\begin{aligned} J\dot{\vec{\omega}} &= -\vec{\omega} \times J\vec{\omega} - k\vec{\omega} + \vec{d} \times \vec{d_0}, \\ \dot{\vec{d_0}} &= -\vec{\omega} \times \vec{d_0}. \end{aligned}$$

From the second equation, we see that equilibrium points must be such that $\vec{\omega} \times \vec{d_0} = 0$, i.e., $\vec{\omega}$ must be parallel to $\vec{d_0}$, or $\vec{\omega} = c\vec{d_0}$, for some scalar c. Rewriting the first equation using this condition, we find

$$J\dot{\vec{\omega}} = -c^2\vec{d_0} \times J\vec{d_0} - ck\vec{d_0} + \vec{d} \times \vec{d_0}.$$

From the above equation, we see that since $-c^2\vec{d_0} \times J\vec{d_0} + \vec{d} \times \vec{d_0}$ is always orthogonal to $\vec{d_0}$, $\dot{\vec{\omega}}$ can be zero only if $c = 0$, i.e., if $\vec{\omega} = 0$. Moreover, we require that $\vec{d} \times \vec{d_0} = 0$, i.e., $\vec{d_0} = \pm\vec{d}$.

Summarizing, we have two equilibria, one with the spacecraft pointing directly at the star, and one with the spacecraft pointing in the opposite direction.

The given function V is indeed a good candidate to study the stability of the "good" equilibrium $\vec{d_0} = \vec{d}$: is it always non-negative, and is equal to zero only at the equilibrium point $(\vec{d_0}, \omega) = (\vec{d}, 0)$. Let us study its time derivative. We get:

$$\dot{V}(\vec{\omega}, \vec{d_0}) = \vec{\omega} \cdot J\dot{\vec{\omega}} + (\vec{d} - \vec{d_0}) \cdot \dot{\vec{d_0}} = \vec{\omega} \cdot (-\vec{\omega} \times J\vec{\omega} - k\vec{\omega} + \vec{d} \times \vec{d_0}) + (\vec{d} - \vec{d_0}) \cdot (-\vec{\omega} \times \vec{d_0}).$$

Using the following property of the triple vector product

$$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u),$$

and remembering that the cross product of two parallel vector is zero, we an simplify the expression of $\dot{V}(\vec{\omega}, \vec{d}_0)$ *as*

$$\dot{V}(\vec{\omega}, \vec{d}_0) = -k|\vec{\omega}|^2,$$

i.e., $\dot{V}$ *is negative semi-definite. As a consequence, Lyapunov's theorem only tells us that the desired equilibrium is stable* in the sense of Lyapunov.

Asymptotic stability can be proven using Lasalle's theorem, since:

- *The set in which* $\dot{V} = 0$ *is the set for which* $(\vec{\omega}, \vec{d}_0) = (0, \cdot)$*, i.e., any state in which the angular velocity is zero—with arbitrary attitude.*
- *However, in case* $\vec{d}_0 \neq \vec{d}$*, we would have* $J\dot{\vec{\omega}} \neq 0$*, thus showing that the largest invariant set in which* $\dot{V} = 0$ *is just the equilibrium* $(\vec{\omega}, \vec{d}_0) = (0, 0)$*.*

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