

## Lecture 19

Last time:

$$\int_{-\infty}^{\infty} w_0(\tau_3)R_{ii}(\tau_1 - \tau_3)d\tau_3 - \int_{-\infty}^{\infty} w_D(\tau_3)R_{is}(\tau_1 - \tau_3)d\tau_3 = 0 \text{ for } \tau_1 \geq 0$$

### Solution in the Free Configuration, Non-Real-Time Filter Case

The applicable condition in this important case is:

$$\int_{-\infty}^{\infty} w_0(\tau_3)R_{ii}(\tau_1 - \tau_3)d\tau_3 - \int_{-\infty}^{\infty} w_D(\tau_3)R_{is}(\tau_1 - \tau_3)d\tau_3 = 0$$

for all  $\tau_1$ .

$$\int_{-\infty}^{\infty} [ \ ] e^{-s\tau_1} d\tau_1 = 0$$

Since this function of  $\tau_1$  must be zero for all  $\tau_1$ , its transform must also be zero.

$$\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3)R_{ii}(\tau_1 - \tau_3)e^{-s(\tau_1 - \tau_3)}e^{-s\tau_3} - \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3)R_{is}(\tau_1 - \tau_3)e^{-s(\tau_1 - \tau_3)}e^{-s\tau_3} = 0$$

$$H_0(s)S_{ii}(s) - D(s)S_{is}(s) = 0$$

$$H_0(s) = \frac{D(s)S_{is}(s)}{S_{ii}(s)}$$

with

$$D(s) = \text{desired signal transfer}$$

$$S_{is}(s) = S_{ss}(s) + S_{ns}(s)$$

$$S_{ii}(s) = S_{ss}(s) + S_{sn}(s) + S_{ns}(s) + S_{nn}(s)$$

This is the optimum filter "transfer function." Since it is non-real-time, it is usually implemented as a weighting sequence in a digital computer. The continuous weighting function is

$$w_0(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} H_0(s)e^{st} ds$$

I will give an integral form which is useful in evaluating this integral.  $H_0(s)$  typically has poles in both left and right hand planes. This does not imply instability in the case of a two-sided transform.

$$o(t) = \int_{-\infty}^{\infty} i(\tau)w_0(t-\tau)d\tau$$

In terms of  $s$ , the integrals are written:

1. If the real part of  $a$  is positive,

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{e^{st}}{(s+a)^n} ds = \begin{cases} \frac{1}{(n-1)!} t^{n-1} e^{-at} & (t > 0) \\ 0 & (t < 0) \end{cases}$$

2. If the real part of  $a$  is negative,

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{e^{st}}{(s+a)^n} ds = \begin{cases} \frac{-1}{(n-1)!} t^{n-1} e^{-at} & (t < 0) \\ 0 & (t > 0) \end{cases}$$

In writing power density spectra, which are rational functions of  $\omega^2$  - if rational at all - we note that  $\omega^2$  is replaced by  $-s^2$ .

Note that the inverse transform of  $H_0(s)$  can be done by expanding  $H_0(s)$  in a partial fraction expansion and integrating each term separately.

This expended problem is more general than the case treated in the text for two reasons:

1. A more general expression for the desired output is permitted
2. Non minimum phase fixed parts of the system,  $F(s)$ , may be handled without stability problems due to cancellation of unstable modes.

### Solution in the Real Time case

It now remains to solve this integral equation for the optimum weighting function,  $w_0(t)$ . As in the case of other operations with linear, invariant systems, the use of transformed functions is most convenient. We define

$$l(\tau_1) = \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4) \\ - \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) R_{is}(\tau_1 + \tau_2 - \tau_3)$$

and we wish to find  $w_0(t)$  to render  $l(\tau_1) = 0$  for  $\tau_1 \geq 0$ .

$$\begin{aligned}
 L(s) &= \int_{-\infty}^{\infty} l(\tau_1) e^{-s\tau_1} d\tau_1 \\
 &= \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) e^{s\tau_2} \int_{-\infty}^{\infty} d\tau_3 w_0(\tau_3) e^{-s\tau_3} \int_{-\infty}^{\infty} d\tau_4 w_F(\tau_4) e^{-s\tau_4} \int_{-\infty}^{\infty} d\tau_1 R_{ii}(\tau_1 + \tau_2 - \tau_3 - \tau_4) e^{-s(\tau_1 + \tau_2 - \tau_3 - \tau_4)} \\
 &\quad - \int_{-\infty}^{\infty} d\tau_2 w_F(\tau_2) e^{s\tau_2} \int_{-\infty}^{\infty} d\tau_3 w_D(\tau_3) e^{-s\tau_3} \int_{-\infty}^{\infty} d\tau_1 R_{is}(\tau_1 + \tau_2 - \tau_3) e^{-s(\tau_1 + \tau_2 - \tau_3)} \\
 &= F(-s)H_0(s)F(s)S_{ii}(s) - F(-s)D(s)S_{is}(s)
 \end{aligned}$$

$$l(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} L(s) e^{s\tau} ds$$

$$e^{s\tau} = e^{(\sigma + j\omega)\tau} = e^{\sigma\tau} e^{j\omega\tau}$$

If  $L(s)$  varies as  $\frac{1}{s}$  for large  $s$ , the line integral is equal to the contour integral around the LHP for  $\tau > 0$  due to the convergence of  $e^{\sigma\tau}$  if  $\sigma$  and  $\tau$  are of opposite sign. Also in that case the line integral is equal to the negative of the contour integral around the RHP for  $\tau < 0$ .

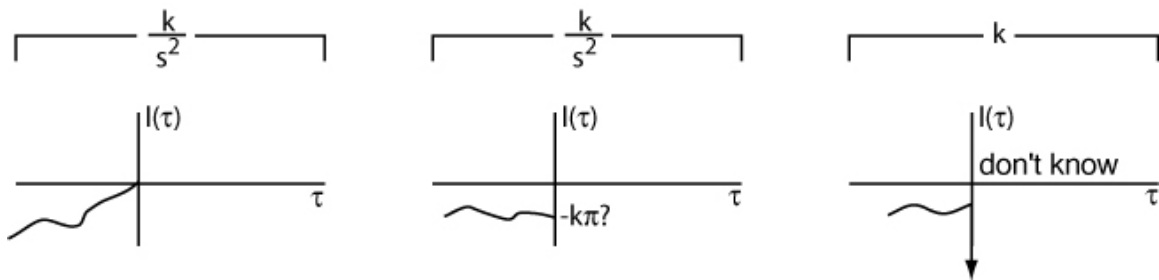
May as well use the change of variable  $s = j\omega$  immediately.

$$L(s) = \int_{-\infty}^{\infty} l(\tau_1) e^{-s\tau_1} d\tau_1$$

If  $L(s)$  is analytic in LHP, and decreasing at least as fast as  $\frac{1}{s}$  for large  $s$ , then  $l(\tau) = 0$  for ( $\tau > 0$ ).

If  $L(s)$  is analytic in RHP, then  $l(\tau) = 0$  for ( $\tau < 0$ ).

If  $L(s)$  is analytic in LHP, and varies for large  $|s|$  as:



So  $l(\tau) = 0$  for  $(\tau > 0)$  and finite for  $(\tau = 0)$  implies  $L(s)$  must be analytic in LHP and decrease at least as fast as  $\frac{1}{s}$  for large  $s$ .

We now take the step called “spectrum factorization”. If the input spectrum is rational – as is or can be always the case in our work – it is clear that the numerator and denominator polynomials in  $s^2$  can be factored into terms which display poles and zeroes in symmetric quads in the  $s$  plane, except for pairs on the imaginary axis. If the numerator and denominator factors which give poles and zeroes in the LHP are multiplied together, this gives the function we defined to be  $S_{ii}(s)_L$ . The remaining factors multiplied together give  $S_{ii}(s)_R$  it has all its poles and zeroes in the RHP. If constants are associated equally with  $S_{ii}(s)$  and  $S_{ii}(s)_R$ , then  $S_{ii}(s)_L = S_{ii}(-s)_R$ . This is clear from the fact that the poles and zeroes of  $S_{ii}(s)_L$  are the negatives of the poles and zeroes of  $S_{ii}(s)_R$ .

$$S_{ii}(s) = S_{ii}(s)_L S_{ii}(s)_R$$

Also, if  $F(s)$  is rational, we can collect those factors defining poles and zeroes in the LHP into  $F(s)_L$  and those factors defining poles and zeroes in the RHP into  $F(s)_R$ . If  $F(s)$  is not rational, it is not clear that factorization can be done since it is almost never a real-valued function. This eliminates from our consideration fixed parts which include, for example, a time delay unless we first approximate the time delay by a non minimum phase rational transfer function. This can be done to any degree of accuracy by including more factors.

For rational  $F(s)$  then, it is factored into

$$F(s) = F(s)_L F(s)_R$$

and in the same way

$$F(-s) = F(-s)_L F(-s)_R$$

The transform exists only for stable functions. So if  $F(s)$  has a pole on the  $j\omega$  axis, treat it as the limit of a LHP pole.

$$\frac{1}{s} \rightarrow \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{1}{s + \varepsilon}$$

$$L(s) = F(-s)_L F(-s)_R H_0(s) F(s)_L F(s)_R S_{ii}(s)_L S_{ii}(s)_R - F(-s)_L F(-s)_R D(s) S_{is}(s)$$

$$\frac{L(s)}{F(s)_R F(-s)_R S_{ii}(s)_R} = F(s)_L F(-s)_L S_{ii}(s)_L H_0(s) - \frac{D(s) F(-s)_L S_{is}(s)}{F(s)_R S_{ii}(s)_R}$$

We are looking for the solution  $l(t) = 0$  for  $t \geq 0$ ; thus  $L(s)$  must be analytic in LHP. But by definition, the denominator functions have no zeroes in LHP so the entire function is analytic in LHP. Its inverse transform must vanish for  $t > 0$ . The same must then be true of the right hand side of the equation.

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)_L F(-s)_L S_{ii}(s)_L H_0(s) e^{st} ds = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{D(s)F(-s)_L S_{is}(s)}{F(s)_R S_{ii}(s)_R} e^{st} ds, \text{ for } t > 0$$

Now if  $H_0(s)$  is to be a stable system - as it must be to minimize the mean squared error - it will be analytic in RHP. Also, by definition, the other functions in the integrand of the left hand side of this equation are analytic in RHP. The left hand integral is then a function of  $t$ , call it  $g(t)$ , which is zero for  $t < 0$ .

$$g(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)_L F(-s)_L S_{ii}(s)_L H_0(s) e^{st} ds = 0, \text{ for } t < 0$$

Then the direct transform of  $g(t)$  would yield the transformed function

$$\begin{aligned} F(s)_L F(-s)_L S_{ii}(s)_L H_0(s) &= \int_{-\infty}^{\infty} g(t) e^{-st} dt \\ &= \int_0^{\infty} g(t) e^{-st} dt \\ &= \frac{1}{2\pi j} \int_0^{\infty} dt e^{-st} \int_{-j\infty}^{j\infty} F(p)_L F(-p)_L S_{ii}(p)_L H_0(p) e^{pt} dp \end{aligned}$$

In this integral,  $t$  takes only positive values; but in this range of  $t$ , the second integral can be replaced by its equal from the former equation.

$$H_0(s) = \frac{1}{2\pi j F(s)_L F(-s)_L S_{ii}(s)_L} \int_0^{\infty} dt e^{-st} \int_{-j\infty}^{j\infty} \frac{D(p)F(-p)_L S_{is}(p)}{F(p)_R S_{ii}(p)_R} e^{pt} dp$$

This is a closed expression for the optimum system function.

In the usual case,  $F(s)$  is a stable, minimum phase function. In that case,  $F(s)_L = F(s)$ ,  $F(s)_R = 1$ ; that is, all the poles and zeroes of  $F(s)$  are in the LHP. Similarly,  $F(-s)_L = 1$ .

Then

$$H_0(s) = \left[ \frac{D(s)S_{is}(s)}{S_{ii}(s)_R} \right]_L \frac{1}{F(s)S_{ii}(s)_L}$$

Thus in this case the optimum transfer function from input to output is

$$F(s)H_0(s) = \frac{\left[ \frac{D(s)S_{is}(s)}{S_{ii}(s)_R} \right]_L}{S_{ii}(s)_L}$$

and the optimum function to be cascaded with the fixed part is obtained from this by division by  $F(s)$ , so that the fixed part is compensated out by cancellation.