16.346 Astrodynamics
Fall 2008

Linearization of the Equations of Motion

- **Deviations from reference** $\mathbf{r}(t) = \mathbf{r}_{ref}(t) + \boldsymbol{\delta}(t) \qquad \mathbf{v}(t) = \mathbf{v}_{ref}(t) + \boldsymbol{\nu}(t)$
- **Equations of motion**

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \mathbf{v} & \frac{d\mathbf{r}_{ref}}{dt} &= \mathbf{v}_{ref} & & & \frac{d\boldsymbol{\delta}}{dt} &= \boldsymbol{\nu} \\ \frac{d\mathbf{v}}{dt} &= \mathbf{g}(\mathbf{r}) & \frac{d\mathbf{v}_{ref}}{dt} &= \mathbf{g}(\mathbf{r}_{ref}) & &\Longrightarrow & \frac{d\boldsymbol{\nu}}{dt} &= \mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r}_{ref}) = \mathbf{G}(\mathbf{r}_{ref})\boldsymbol{\delta} = \mathbf{G}(t)\boldsymbol{\delta} \end{aligned}$$

since $$\mathbf{g}(\mathbf{r}) = \mathbf{g}(\mathbf{r}_{ref}) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{r}}\right|_{\mathbf{r}=\mathbf{r}_{ref}} \boldsymbol{\delta} + \cdots = \mathbf{g}(\mathbf{r}_{ref}) + \mathbf{G}(\mathbf{r}_{ref})\boldsymbol{\delta} + O(\delta^2)$$

- **State vector representation**

$$\frac{d}{dt}\underbrace{\begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\nu} \end{bmatrix}}_{\mathbf{x}(t)} = \underbrace{\begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{G}(t) & \mathbf{O} \end{bmatrix}}_{\mathbf{F}(t)} \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\nu} \end{bmatrix} \qquad \text{or} \qquad \boxed{\frac{d\mathbf{x}}{dt} = \mathbf{F}(t)\mathbf{x}}$$

The State Transition Matrix

- **Define**

$$\begin{aligned} \boldsymbol{\Phi}(t, t_0) &= \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_6(t) \end{bmatrix} \\ \boldsymbol{\Phi}(t_0, t_0) &= \begin{bmatrix} \mathbf{x}_1(t_0) & \mathbf{x}_2(t_0) & \ldots & \mathbf{x}_6(t_0) \end{bmatrix} = \mathbf{I} \end{aligned}$$

- **Matrix differential equation**

$$\frac{d}{dt}\boldsymbol{\Phi}(t, t_0) = \mathbf{F}(t)\boldsymbol{\Phi}(t, t_0) \qquad \text{with} \qquad \boldsymbol{\Phi}(t_0, t_0) = \mathbf{I}$$

- **Fundamental property** $\boxed{\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\mathbf{x}(t_0)}$

Symplectic Matrices

- **Definition**

  An even-dimensional matrix $\mathbf{A}$ is symplectic if

$$\boxed{\mathbf{A}^{\mathsf{T}}\mathbf{J}\mathbf{A} = \mathbf{J}} \qquad \text{where} \qquad \mathbf{J} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{bmatrix}$$

Note: Since $\mathbf{J}^2 = -\mathbf{I}$, the $\mathbf{J}$ matrix is analogous to the imaginary $\sqrt{-1}$ in complex algebra.

- **Inverse of a Symplectic Matrix**

  Postmultiply by $\mathbf{A}^{-1}$ and premultiply by $\mathbf{J}$ to obtain

$$\boxed{\mathbf{A}^{-1} = -\mathbf{J}\mathbf{A}^{\mathsf{T}}\mathbf{J}}$$

The State Transition Matrix is Symplectic

- **Symplectic Property of $\boldsymbol{\Phi}(t, t_0)$**

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{\Phi}^{\mathrm{T}}(t,t_0)\mathbf{J}\boldsymbol{\Phi}(t,t_0) &= \frac{d\boldsymbol{\Phi}^{\mathrm{T}}}{dt}\mathbf{J}\boldsymbol{\Phi} + \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{J}\frac{d\boldsymbol{\Phi}}{dt} \\
&= \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{J}\boldsymbol{\Phi} + \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{J}\mathbf{F}\boldsymbol{\Phi} \\
&= \boldsymbol{\Phi}^{\mathrm{T}}[\mathbf{F}^{\mathrm{T}}\mathbf{J} + \mathbf{J}\mathbf{F}]\boldsymbol{\Phi} \\
&= \boldsymbol{\Phi}^{\mathrm{T}}\begin{bmatrix} \mathbf{G}(t) - \mathbf{G}^{\mathrm{T}}(t) & \mathbf{O} \\ \mathbf{O} & \mathbf{I} - \mathbf{I} \end{bmatrix}\boldsymbol{\Phi} \\
&= \mathbf{O}
\end{aligned}
$$

  since the gravity-gradient matrix $\mathbf{G} = \mathbf{G}^{\mathrm{T}}$ is symmetric. Therefore

$$
\boldsymbol{\Phi}^{\mathrm{T}}(t,t_0)\mathbf{J}\boldsymbol{\Phi}(t,t_0) = \text{constant} = \boldsymbol{\Phi}^{\mathrm{T}}(t_0,t_0)\mathbf{J}\boldsymbol{\Phi}(t_0,t_0) = \mathbf{I}^{\mathrm{T}}\mathbf{J}\mathbf{I} = \mathbf{J}
$$

- **Inverse of $\boldsymbol{\Phi}(t, t_0)$**

  From the partitions
$$
\boxed{\boldsymbol{\Phi}(t,t_0) = \begin{bmatrix} \boldsymbol{\Phi}_1(t,t_0) & \boldsymbol{\Phi}_2(t,t_0) \\ \boldsymbol{\Phi}_3(t,t_0) & \boldsymbol{\Phi}_4(t,t_0) \end{bmatrix}}
$$

  the inverse is

$$
\boxed{\boldsymbol{\Phi}^{-1}(t,t_0) = \boldsymbol{\Phi}(t_0,t) = \begin{bmatrix} \boldsymbol{\Phi}_4^{\mathrm{T}}(t,t_0) & -\boldsymbol{\Phi}_2^{\mathrm{T}}(t,t_0) \\ -\boldsymbol{\Phi}_3^{\mathrm{T}}(t,t_0) & \boldsymbol{\Phi}_1^{\mathrm{T}}(t,t_0) \end{bmatrix}}
$$

Fundamental Perturbation Matrices

For the discussion of linear deviations from a reference orbit, define

$$
\mathbf{x}(t) = \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_0 = \begin{bmatrix} \delta\mathbf{r}_0 \\ \delta\mathbf{v}_0 \end{bmatrix}
$$

so that

$$
\begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} = \boldsymbol{\Phi}(t,t_0)\begin{bmatrix} \delta\mathbf{r}_0 \\ \delta\mathbf{v}_0 \end{bmatrix} \quad \text{where} \quad \boldsymbol{\Phi}(t,t_0) = \begin{bmatrix} \dfrac{\partial\mathbf{r}}{\partial\mathbf{r}_0} & \dfrac{\partial\mathbf{r}}{\partial\mathbf{v}_0} \\ \dfrac{\partial\mathbf{v}}{\partial\mathbf{r}_0} & \dfrac{\partial\mathbf{v}}{\partial\mathbf{v}_0} \end{bmatrix}_{ref} = \begin{bmatrix} \widetilde{\mathbf{R}}(t) & \mathbf{R}(t) \\ \widetilde{\mathbf{V}}(t) & \mathbf{V}(t) \end{bmatrix}
$$

$$
\begin{bmatrix} \delta\mathbf{r}_0 \\ \delta\mathbf{v}_0 \end{bmatrix} = \boldsymbol{\Phi}(t_0,t)\begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} \quad \text{where} \quad \boldsymbol{\Phi}(t_0,t) = \begin{bmatrix} \dfrac{\partial\mathbf{r}_0}{\partial\mathbf{r}} & \dfrac{\partial\mathbf{r}_0}{\partial\mathbf{v}} \\ \dfrac{\partial\mathbf{v}_0}{\partial\mathbf{r}} & \dfrac{\partial\mathbf{v}_0}{\partial\mathbf{v}} \end{bmatrix}_{ref} = \begin{bmatrix} \mathbf{V}^{\mathrm{T}}(t) & -\mathbf{R}^{\mathrm{T}}(t) \\ -\widetilde{\mathbf{V}}^{\mathrm{T}}(t) & \widetilde{\mathbf{R}}^{\mathrm{T}}(t) \end{bmatrix}
$$

since $\qquad \boldsymbol{\Phi}(t,t_0)\boldsymbol{\Phi}(t_0,t) = \mathbf{I} \quad \text{and} \quad \boldsymbol{\Phi}(t_0,t) = \boldsymbol{\Phi}^{-1}(t,t_0)$

Navigation Matrix $\quad \boldsymbol{\Phi}(t, t_0) \quad$ **Guidance Matrix** $\quad \boldsymbol{\Phi}(t, t_1)$

Let $t_0 \le t \le t_1$ and define

$$\boldsymbol{\Phi}(t, t_0) = \begin{bmatrix} \widetilde{\mathbf{R}}(t) & \mathbf{R}(t) \\ \widetilde{\mathbf{V}}(t) & \mathbf{V}(t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Phi}(t, t_1) = \begin{bmatrix} \widetilde{\mathbf{R}}^\star(t) & \mathbf{R}^\star(t) \\ \widetilde{\mathbf{V}}^\star(t) & \mathbf{V}^\star(t) \end{bmatrix}$$

Then

$$\frac{d\mathbf{R}(t)}{dt} = \mathbf{V}(t) \qquad \mathbf{R}(t_0) = \mathbf{O} \qquad \frac{d\mathbf{R}^\star(t)}{dt} = \mathbf{V}^\star(t) \qquad \mathbf{R}^\star(t_1) = \mathbf{O}$$

$$\frac{d\mathbf{V}(t)}{dt} = \mathbf{G}(t)\mathbf{R}(t) \qquad \mathbf{V}(t_0) = \mathbf{I} \qquad \frac{d\mathbf{V}^\star(t)}{dt} = \mathbf{G}(t)\mathbf{R}^\star(t) \qquad \mathbf{V}^\star(t_1) = \mathbf{I}$$

The differential equations for the "tilde" matrices are the same but with initial conditions

$$\widetilde{\mathbf{R}}(t_0) = \mathbf{I} \qquad \widetilde{\mathbf{R}}^\star(t_1) = \mathbf{I}$$

$$\widetilde{\mathbf{V}}(t_0) = \mathbf{O} \qquad \widetilde{\mathbf{V}}^\star(t_1) = \mathbf{O}$$

Furthermore,

$$\widetilde{\mathbf{C}}^\star = \widetilde{\mathbf{V}}^\star \widetilde{\mathbf{R}}^{\star -1} = \left.\frac{\partial \mathbf{v}}{\partial \mathbf{r}}\right|_{\mathbf{v_1}=\text{constant}} \qquad \mathbf{C}^\star = \mathbf{V}^\star \mathbf{R}^{\star -1} = \left.\frac{\partial \mathbf{v}}{\partial \mathbf{r}}\right|_{\mathbf{r_1}=\text{constant}}$$

Hence

$$\left.\boldsymbol{\Phi}(t, t_1)\right|_{t=t_0} = \left.\boldsymbol{\Phi}^{-1}(t, t_0)\right|_{t=t_1} \quad \Longrightarrow \quad \begin{bmatrix} \widetilde{\mathbf{R}}^\star(t_0) & \mathbf{R}^\star(t_0) \\ \widetilde{\mathbf{V}}^\star(t_0) & \mathbf{V}^\star(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{\mathrm{T}}(t_1) & -\mathbf{R}^{\mathrm{T}}(t_1) \\ -\widetilde{\mathbf{V}}^{\mathrm{T}}(t_1) & \widetilde{\mathbf{R}}^{\mathrm{T}}(t_1) \end{bmatrix}$$

Differential Equation for the $\mathbf{C}^\star$ Matrix

$$\mathbf{C}^\star \mathbf{R}^\star = \mathbf{V}^\star \qquad \text{and} \qquad \mathbf{C}^{\star -1} = \mathbf{R}^\star \mathbf{V}^{\star -1} \qquad \text{or} \qquad \mathbf{C}^{\star -1}\mathbf{V}^\star = \mathbf{R}^\star$$

Differentiate the first expression:

$$\frac{d\mathbf{C}^\star}{dt}\mathbf{R}^\star + \mathbf{C}^\star \frac{d\mathbf{R}^\star}{dt} = \frac{d\mathbf{V}^\star}{dt} \quad \Longrightarrow \quad \frac{d\mathbf{C}^\star}{dt}\mathbf{R}^\star + \mathbf{C}^\star \mathbf{V}^\star = \mathbf{G}\mathbf{R}^\star$$

Finally, postmultiply by $\mathbf{R}^{\star -1}$ to obtain

$$\frac{d\mathbf{C}^\star}{dt} + \mathbf{C}^{\star 2} = \mathbf{G}$$

Since $\mathbf{G}$ is symmetric, then $\mathbf{C}^\star$ is symmetric. Because $\mathbf{C}^\star(t_1)$ is infinite, it is better to use the equation for the inverse matrix.

$$\frac{d\mathbf{C}^{\star -1}}{dt}\mathbf{V}^\star + \mathbf{C}^{\star -1}\frac{d\mathbf{V}^\star}{dt} = \frac{d\mathbf{R}^\star}{dt} \quad \Longrightarrow \quad \frac{d\mathbf{C}^{\star -1}}{dt}\mathbf{V}^\star + \mathbf{C}^{\star -1}\mathbf{G}\mathbf{R}^\star = \mathbf{V}^\star$$

Hence

$$\frac{d\mathbf{C}^{\star -1}}{dt} + \mathbf{C}^{\star -1}\mathbf{G}\mathbf{C}^{\star -1} = \mathbf{I} \qquad \text{with} \qquad \mathbf{C}^{\star -1}(t_1) = \mathbf{O}$$

Note: For the constant gravity case we have $\mathbf{G} = \mathbf{O}$ so that $\mathbf{C}^{\star -1} = (t - t_1)\mathbf{I}$.

- **Fixed-Time-of-Arrival Correction**

$$\delta\mathbf{x}(t) = \mathbf{\Phi}(t,t_1)\,\delta\mathbf{x}(t_1) \quad \Longrightarrow \quad \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{R}}^\star(t) & \mathbf{R}^\star(t) \\ \widetilde{\mathbf{V}}^\star(t) & \mathbf{V}^\star(t) \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \delta\mathbf{v}(t_A) \end{bmatrix}$$

Hence

$$\delta\mathbf{r}(t) = \mathbf{R}^\star(t)\,\delta\mathbf{v}(t_A)$$
$$\delta\mathbf{v}(t) = \mathbf{V}^\star(t)\,\delta\mathbf{v}(t_A)$$

Eliminate $\delta\mathbf{v}(t_A)$

$$\delta\mathbf{v}(t) = \mathbf{V}^\star(t)\mathbf{R}^{\star\,-1}(t)\,\delta\mathbf{r}(t) = \mathbf{C}^\star(t)\,\delta\mathbf{r}(t)$$

Velocity correction $\quad \Delta\mathbf{v}(t) = \delta\mathbf{v}(t^+) - \delta\mathbf{v}(t^-) = \mathbf{C}^\star(t)\,\delta\mathbf{r}(t) - \delta\mathbf{v}(t^-)$

- **Variable-Time-of-Arrival Correction**

$$\begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{R}}^\star(t) & \mathbf{R}^\star(t) \\ \widetilde{\mathbf{V}}^\star(t) & \mathbf{V}^\star(t) \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t_A) \\ \delta\mathbf{v}(t_A) \end{bmatrix}$$

Multiply through by $\begin{bmatrix} -\mathbf{C}^\star(t) & \mathbf{I} \end{bmatrix}$

$$\begin{bmatrix} -\mathbf{C}^\star(t) & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{C}^\star(t) & \mathbf{I} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{R}}^\star(t) & \mathbf{R}^\star(t) \\ \widetilde{\mathbf{V}}^\star(t) & \mathbf{V}^\star(t) \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t_A) \\ \delta\mathbf{v}(t_A) \end{bmatrix}$$

Hence, [using the starred form of Eqs. (9.57)], we have

$$-\mathbf{C}^\star(t)\,\delta\mathbf{r}(t) + \delta\mathbf{v}(t) = \underbrace{\left[-\mathbf{C}^\star(t)\widetilde{\mathbf{R}}^\star(t) + \widetilde{\mathbf{V}}^\star(t)\right]}_{= -\mathbf{R}^{\star\,-\mathrm{T}}(t)}\,\delta\mathbf{r}(t_A)$$

or

$$\delta\mathbf{v}(t) = \mathbf{C}^\star(t)\,\delta\mathbf{r}(t) - \mathbf{R}^{\star\,-\mathrm{T}}(t)\,\delta\mathbf{r}(t_A)$$

1. Choosing $\delta\mathbf{r}(t_A)$

$$\mathbf{r}_p(t_A + \delta t) = \mathbf{r}_p(t_A) + \mathbf{v}_p(t_A)\,\delta t$$
$$\mathbf{r}(t_A + \delta t) = \mathbf{r}(t_A) + \mathbf{v}(t_A)\,\delta t$$

Then $\quad \mathbf{r}(t_A + \delta t) = \mathbf{r}_p(t_A + \delta t) \quad \Longrightarrow \quad \delta\mathbf{r}(t_A) = \mathbf{r}(t_A) - \mathbf{r}_p(t_A) = -\mathbf{v}_r(t_A)\,\delta t$

where $\quad \mathbf{v}_r(t_A) = \mathbf{v}(t_A) - \mathbf{v}_p(t_A)$. Hence

$$\delta\mathbf{v}(t) = \mathbf{C}^\star(t)\,\delta\mathbf{r}(t) + \underbrace{\mathbf{R}^{\star\,-\mathrm{T}}(t)\mathbf{v}_r(t_A)}_{= \mathbf{w}(t)}\,\delta t \qquad \text{or} \qquad \Delta\mathbf{v}'(t) = \Delta\mathbf{v}(t) + \mathbf{w}(t)\,\delta t$$

2. Choosing δt to minimize $|\Delta\mathbf{v}'(t)|$

$$\delta t_A = -\frac{\Delta\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}} \qquad \text{and} \qquad \text{Min}\,\Delta\mathbf{v}' = \underbrace{\left(\mathbf{I} - \frac{\mathbf{w}\mathbf{w}^\mathrm{T}}{\mathbf{w}^\mathrm{T}\mathbf{w}}\right)}_{\text{Projection operator }\mathbf{M}}\Delta\mathbf{v} = \mathbf{M}\,\Delta\mathbf{v}$$