16.346 Astrodynamics
Fall 2008

## Jacobi's Equations

If the third mass $m_3 \equiv m$ is infinitesimal, then

$$m\Big[\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\Big] = -\frac{Gmm_1}{\rho_1^3}\boldsymbol{\rho}_1 - \frac{Gmm_2}{\rho_2^3}\boldsymbol{\rho}_2$$

where

$$\boldsymbol{\rho}_1 = \mathbf{r} - \mathbf{r}_1 \qquad \boldsymbol{\rho}_2 = \mathbf{r} - \mathbf{r}_2 \qquad \mathbf{r} = \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \xi\,\mathbf{i}_\xi + \eta\,\mathbf{i}_\eta + \zeta\,\mathbf{i}_\zeta$$

$$\boldsymbol{\omega} = \omega\,\mathbf{i}_\zeta \quad \text{with} \quad \omega^2 = \frac{G(m_1 + m_2 + m)}{r_{12}^3} \approx \frac{G(m_1 + m_2)}{r_{12}^3}$$

With $m_1$ and $m_2$ on $\xi$-axis, then

$$\begin{aligned} \mathbf{r}_1 &= \xi_1\,\mathbf{i}_\xi \qquad & \rho_1^2 &= (\xi - \xi_1)^2 + \eta^2 + \zeta^2 \\ \mathbf{r}_2 &= \xi_2\,\mathbf{i}_\xi \qquad & \rho_2^2 &= (\xi - \xi_2)^2 + \eta^2 + \zeta^2 \end{aligned}$$

Now

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2(\xi\,\mathbf{i}_\xi + \eta\,\mathbf{i}_\eta)$$

so that

$$\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \omega^2(\xi\,\mathbf{i}_\xi + \eta\,\mathbf{i}_\eta) - \frac{Gm_1}{\rho_1^3}\boldsymbol{\rho}_1 - \frac{Gm_2}{\rho_2^3}\boldsymbol{\rho}_2$$

Define

$$\boxed{\mathcal{J}(\xi, \eta, \zeta) = \frac{\omega^2}{2}(\xi^2 + \eta^2) + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2}}$$

Then

$$\boxed{\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \left[\frac{\partial \mathcal{J}}{\partial \mathbf{r}}\right]^{\mathrm{T}}} \qquad \text{or} \qquad \begin{aligned} \frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} &= \frac{\partial \mathcal{J}}{\partial \xi} \\ \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} &= \frac{\partial \mathcal{J}}{\partial \eta} \\ \frac{d^2\zeta}{dt^2} &= \frac{\partial \mathcal{J}}{\partial \zeta} \end{aligned}$$

## Jacobi's Integral

Take scalar product with $d\mathbf{r}/dt$

$$\underbrace{\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt}}_{\frac{1}{2}\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right)} + \underbrace{2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}}_{=0} = \underbrace{\frac{\partial \mathcal{J}}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt}}_{\frac{d\mathcal{J}}{dt}}$$

Integrate to obtain

$$\frac{1}{2}\left(\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+C\right)=\mathcal{J} \qquad \text{or} \qquad v_{rel}^2 = 2\mathcal{J}(\xi,\eta,\zeta)-C$$

Hence

$$\boxed{v_{rel}^2 = \omega^2(\xi^2+\eta^2)+\frac{2Gm_1}{\rho_1}+\frac{2Gm_2}{\rho_2}-C}$$

## Surfaces of Zero Relative Velocity

In $\xi, \eta, \zeta$ space, surfaces of zero relative velocity are

$$\omega^2(\xi^2+\eta^2)+\frac{2Gm_1}{\rho_1}+\frac{2Gm_2}{\rho_2} = \text{constant}$$

In the $\xi, \eta$ plane (in terms of bipolar coordinates) curves of zero relative velocity are

$$Gm_1\left(\frac{\rho_1^2}{\rho^3}+\frac{2}{\rho_1}\right)+Gm_2\left(\frac{\rho_2^2}{\rho^3}+\frac{2}{\rho_2}\right) = \text{constant}$$

**Note:** In the two-body problem, curves of zero relative velocity are

$$v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right) = 0 \qquad \text{or} \qquad r = 2a$$

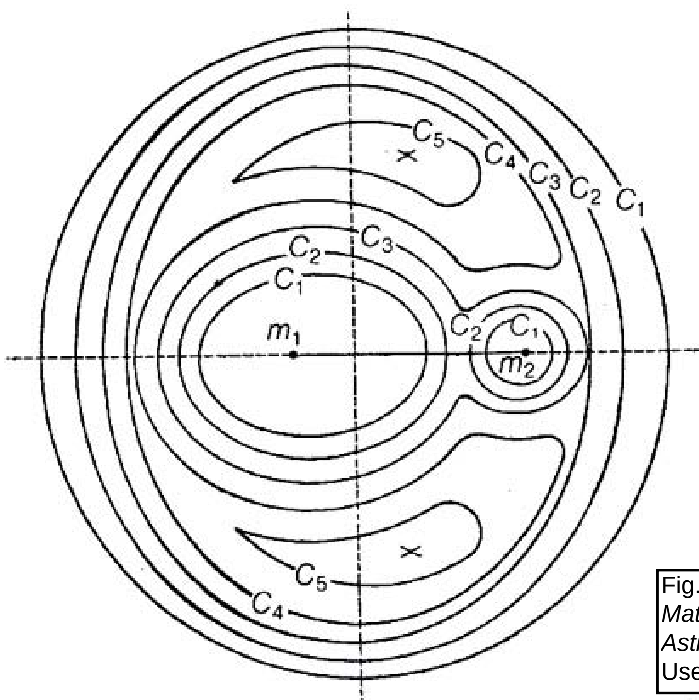


Fig. 8.1 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Jacobi's integral $\left(\frac{d\zeta}{dt}\right)^2 = \frac{4Gm}{\rho} - C$ where $\rho^2 = D^2 + \zeta^2$ and $\omega^2 = \frac{Gm}{4D^3}$

With velocity $v_0$ when $\zeta = 0$, then $C = \frac{4Gm}{D} - v_0^2$.

Hence
$$\left(\frac{d\zeta}{dt}\right)^2 = v_0^2 - 16\omega^2 D^2\left(1 - \frac{D}{\rho}\right)$$

Define $\zeta = D\tan\theta$, $\rho = D\sec\theta$ and $B = \frac{v_0^2}{16\omega^2 D^2}$.

Then
$$\left(\frac{d\theta}{dt}\right)^2 = 16\omega^2\cos^4\theta[B - (1-\cos\theta)]$$

Now $\frac{d\theta}{dt} = 0$ if and only if $B \le 1$. Define $\theta = \theta_m$ when $\frac{d\theta}{dt} = 0$. Then $B = 1 - \cos\theta_m$.

Therefore,
$$\left(\frac{d\theta}{dt}\right)^2 = 16\omega^2\cos^4\theta(\cos\theta - \cos\theta_m)$$

Define $x = \cos\theta$ and $x_m = \cos\theta_m$.

Then
$$\left(\frac{dx}{dt}\right)^2 = 16\omega^2 x^4(1-x^2)(x-x_m)$$

Let $T$ be the quarter period. Then $4\omega T = \int_{x_m}^{1} \frac{dx}{x^2\sqrt{P(x)}}$ with $P(x) = (1-x^2)(x-x_m)$

Integrate by parts
$$4\omega x_m T = \left.\frac{\sqrt{P(x)}}{x}\right|_{x_m}^{1} + \frac{1}{2}\int_{x_m}^{1}\frac{x\,dx}{\sqrt{P(x)}} + \frac{1}{2}\int_{x_m}^{1}\frac{dx}{x\sqrt{P(x)}}$$

Convert $P(x)$ to fourth degree with the substitution $x = 1 - z^2$

$$4\omega x_m T = \int_0^{\alpha}\frac{(1-z^2)\,dz}{\sqrt{Q(z)}} + \int_0^{\alpha}\frac{dz}{(1-z^2)\sqrt{Q(z)}}$$

$$Q(z) = 2\alpha^2\left(1 - \frac{z^2}{2}\right)\left(1 - \frac{z^2}{\alpha^2}\right) \equiv R(y) = (1-y^2)(1-k^2y^2)$$

where we have defined $\alpha = \sqrt{1-x_m}$, $z = \alpha y$ and $k^2 = \frac{1}{2}\alpha^2$ and obtain

$$4\sqrt{2}\,\omega x_m T = \int_0^1 \frac{(1-2k^2y^2)\,dy}{\sqrt{R(y)}} + \int_0^1\frac{dy}{(1-2k^2y^2)\sqrt{R(y)}}$$

After some algebra, convert to Legendre form with $y = \sin\phi$ to obtain

$$\boxed{4\sqrt{2}\,\omega x_m T = 2E(k) - K(k) + \Pi(2k^2, k, \tfrac{1}{2}\pi)}$$