16.346 Astrodynamics
Fall 2008

Variational Equations using the Disturbing Function

Recall the variational equatioms

$$\frac{d\mathbf{s}}{dt} = \frac{\partial \mathbf{s}}{\partial t} + \frac{\partial \mathbf{s}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{F}\,\mathbf{s} + \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_d \end{bmatrix} \qquad \Longrightarrow \qquad \frac{\partial \mathbf{s}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} = \begin{bmatrix} \mathbf{0} \\ \mathbf{a}_d \end{bmatrix}$$

which we can write as

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} &= \mathbf{0} \\ \frac{\partial \mathbf{v}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} &= \mathbf{a}_d \end{aligned} \qquad \text{or} \qquad \begin{aligned} \left[\frac{\partial \mathbf{v}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} &= \mathbf{0} \\ \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} &= \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \mathbf{a}_d \end{aligned}$$

If we use the gradient of the disturbing function R for the disturbing acceleration

$$\boxed{\mathbf{a}_d^{\mathrm{T}} = \frac{\partial R}{\partial \mathbf{r}}}$$

then we have

$$\left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \mathbf{a}_d = \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \left[\frac{\partial R}{\partial \mathbf{r}}\right]^{\mathrm{T}} = \left[\frac{\partial R}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} = \left[\frac{\partial R}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}}$$

so that

$$\underbrace{\left\{\left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\alpha}} - \left[\frac{\partial \mathbf{v}}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}} \frac{\partial \mathbf{r}}{\partial \boldsymbol{\alpha}}\right\}}_{\text{Lagrange Matrix } \mathbf{L}} \frac{d\boldsymbol{\alpha}}{dt} = \left[\frac{\partial R}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}}$$

Therefore, the variational equation using the matrix $\mathbf{L}$ is

$$\boxed{\mathbf{L}\,\frac{d\boldsymbol{\alpha}}{dt} = \left[\frac{\partial R}{\partial \boldsymbol{\alpha}}\right]^{\mathrm{T}}}$$

The Lagrange Matrix is skew-symmetric, i.e., $\mathbf{L} = -\mathbf{L}^{\mathrm{T}}$. Because of the skew-symmetry, there are only 15 elements to calculate and only 6 of these are different from zero.

Lagrange's Planetary Equations

$$\begin{aligned} \frac{d\Omega}{dt} &= \frac{1}{nab\sin i}\frac{\partial R}{\partial i} & \frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\ \frac{di}{dt} &= -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega} & \frac{de}{dt} &= -\frac{b}{na^3e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4e}\frac{\partial R}{\partial \lambda} \\ \frac{d\omega}{dt} &= -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3e}\frac{\partial R}{\partial e} & \frac{d\lambda}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4e}\frac{\partial R}{\partial e} \end{aligned}$$

Effect of the J_2 Term on Satellite Orbits

Gravitational potential function of the earth

$$V(r,\phi) = \frac{Gm}{r} \underbrace{- \frac{Gm}{r}\sum_{k=2}^{\infty} J_k \left(\frac{r_{eq}}{r}\right)^k P_k(\cos\phi)}_{=R} \qquad \textbf{(8.92)}$$

where the angle ϕ is the colatitude with $\cos\phi = \mathbf{i}_r \cdot \mathbf{i}_z$ and

$$\begin{aligned}\mathbf{i}_r = &\ [\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i]\,\mathbf{i}_x \\ &+ [\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i]\,\mathbf{i}_y \\ &+ \sin(\omega+f)\sin i\,\mathbf{i}_z\end{aligned}$$

Problem 3–21

Hence $\cos\phi = \sin(\omega+f)\sin i$ so that

$$R = -\frac{GmJ_2r_{eq}^2}{2p^3}(1+e\cos f)^3[3\sin^2(\omega+f)\sin^2 i - 1] + O[(r_{eq}/r)^3]$$

$$\begin{aligned}\overline{R} &= \frac{1}{2\pi}\int_0^{2\pi} R\,dM \qquad \text{where} \qquad dM = n\,dt \quad \text{and} \quad r^2 df = h\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{n}{h} R r^2 df = \frac{\mu J_2 r_{eq}^2}{4a^3(1-e^2)^{\frac{3}{2}}}(2 - 3\sin^2 i)\end{aligned}$$

Averaged Variational Equations

$$\overline{\frac{d\Omega}{dt}} = \frac{1}{nab\sin i}\frac{\partial\overline{R}}{\partial i} \qquad\qquad \overline{\frac{da}{dt}} = 0$$

$$\overline{\frac{di}{dt}} = 0 \qquad\qquad \overline{\frac{de}{dt}} = 0$$

$$\overline{\frac{d\omega}{dt}} = -\frac{\cos i}{nab\sin i}\frac{\partial\overline{R}}{\partial i} + \frac{b}{na^3e}\frac{\partial\overline{R}}{\partial e} \qquad\qquad \overline{\frac{d\lambda}{dt}} = -\frac{2}{na}\frac{\partial\overline{R}}{\partial a} - \frac{b^2}{na^4e}\frac{\partial\overline{R}}{\partial e}$$

For the Earth

Problem 10–12

$$\overline{\frac{d\Omega}{dt}} = -\frac{3}{2}J_2\left(\frac{r_{eq}}{p}\right)^2 n\cos i = -9.96\left(\frac{r_{eq}}{a}\right)^{3.5}(1-e^2)^{-2}\cos i \quad \text{degrees/day}$$

$$\overline{\frac{d\omega}{dt}} = \frac{3}{4}J_2\left(\frac{r_{eq}}{p}\right)^2 n(5\cos^2 i - 1) = 5.0\left(\frac{r_{eq}}{a}\right)^{3.5}(1-e^2)^{-2}(5\cos^2 i - 1) \quad \text{degrees/day}$$

Coefficients of the earth's gravitational potential ($\times 10^6$)

$$J_2 = 1,082.28 \pm 0.03 \qquad J_5 = -0.2 \pm 0.1$$
$$J_3 = -2.3 \pm 0.2 \qquad J_6 = 1.0 \pm 0.8$$
$$J_4 = -2.12 \pm 0.05$$