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16.423J

Bone Homework Solution Set

PROBLEM 1.

Section modulus of idealized femoral shafts:
We find the section modulus, Z, by the equation $Z = I/r_o$, where $I = 0.25\pi(r_o^4 - r_i^4)$, simplified as

$$Z = (0.25\pi r_o^4 - 0.25\pi i r_i^4)/ r_o \quad (1)$$

For adult males of age 25, Z is **2.90 cm³**.
For adult females of age 25, Z is **1.79 cm³**.

Outer radius, r_o, as a function of changing inner radius, r_i:
As years pass, and a human's osteoblasts and osteoclasts are constantly reforming and absorbing bone. This dynamic process can result in changes in both inner and outer dimensions of the cortical layer of the bone, but bone strength can nevertheless be maintained. If the load on the skeleton remains constant, then the bones can maintain constant strain rate if they maintain a constant section modulus. Consequently, if the inner radius of the cortical layer increases due to bone remodeling, a correspondent increase in outer radius due to bone modeling can prevent a change in bone section modulus. We can see this section modulus maintenance by returning to Equation (1):

$$Z = (0.25\pi r_o^4 - 0.25\pi i r_i^4)/ r_o$$

We set Z to be a constant and allow the inner radius r_i to increase linearly with time. For this problem r_i increases by 0.004 cm/year, a value which we can call r_{rate}. We can collect outer radius, ro, terms on one side and ri terms on the other side of the equation. Then we can solve *iteratively* for the outer radius ro, which is found implicitly in Equation (2). The variable t is the time in years since the starting conditions.

$$\mathbf{0.25\pi r_o^4 - Z r_o = 0.25\pi\,(r_{i,initial} + 0.004t)^4} \quad (2)$$

I used an iterative loop in Matlab to calculate the outer radius for each year between age 25 and age 95, subject to the inner radius increasing by 0.004 cm/year and the section modulus remaining constant (at either 2.90 cm^3 for males or 1.79 cm^3 for females). The Matlab code is attached.

The results show that it is indeed physically possible for the outer radius to increase in such a way that the section modulus remains the same when the inner radius increases. For a female aging from 25 years old to 80 years old, the outer radius increases from 1.40 cm to 1.49 cm to compensate for an inner radius change from 0.90 cm to 1.12 cm. In other words, in 55 years, a 6.4% increase in r_o occurs for a 24% increase in r_i. For males going from age 25 to age 80, a 6.5% increase in r_o occurs for an 18% increase in r_i. Even at age 80, for both females and males, the inner radius is sufficiently smaller than the outer radius as to make physical sense for the bone's cortical thickness: for females, the thickness ($r_o - r_i$) decreases from 0.50 cm at age 25 to 0.37 cm at age 80.

The following figures show the change in cortical layer inner and outer radii with time.

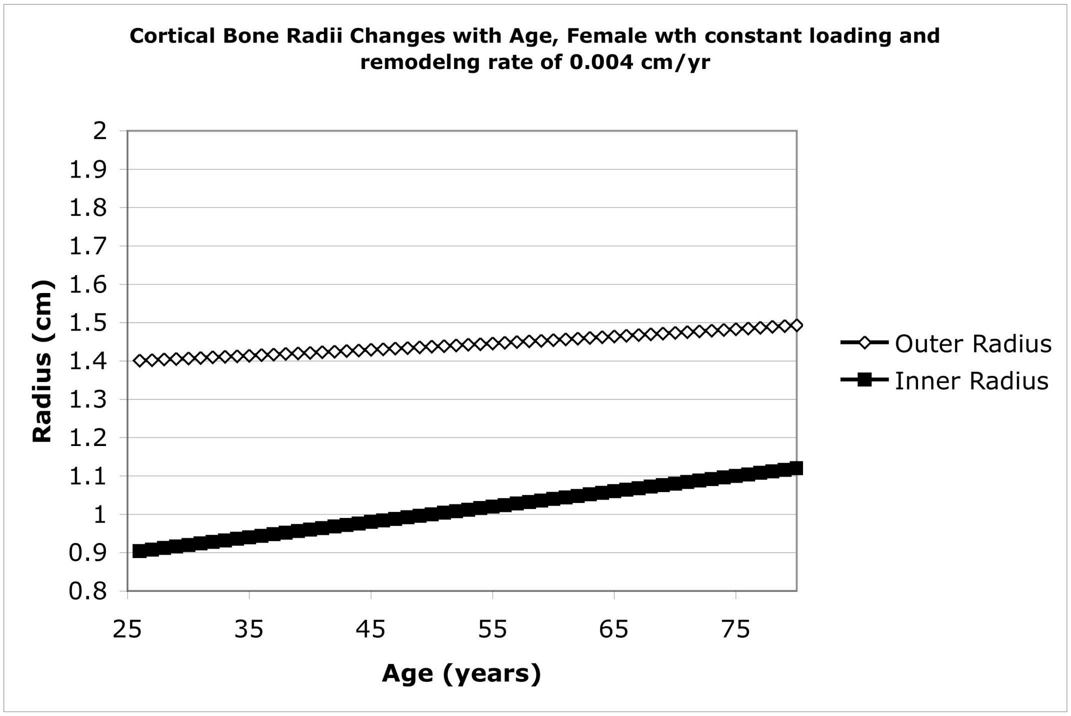

Figure 1: Inner and outer radius changes with time for average female, under constant skeletal loading

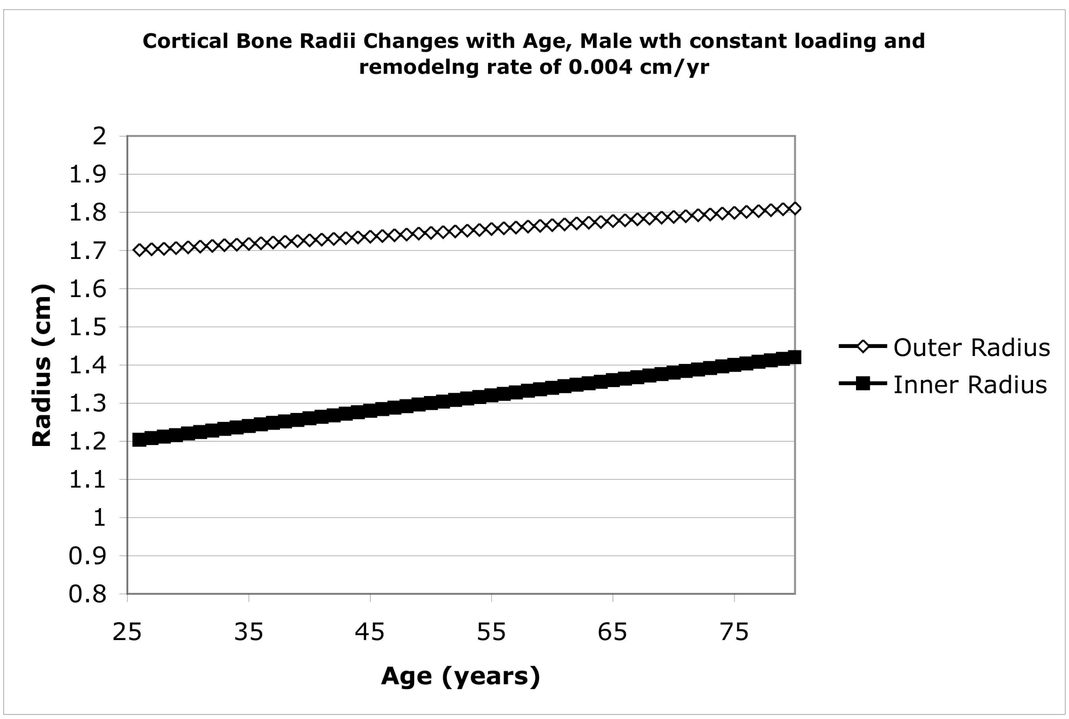

Figure 2: Inner and outer radius changes with time for average male, under constant skeletal loading

Effect on bone mineral density, BMD:

The above figures show that the section modulus can remain constant due to an increasing outer radius as remodeling increases the inner radius. While section modulus is stablilized, the areal mineral density of bone, BMD, does not remain constant. BMD decreases as remodeling takes away from the inside of the cortical layer and modeling adds to the outside. Figure 3 shows the percent change in BMD for females and males subject to the above changes in inner and outer cortical bone radii. At age 80, females have experienced a 20% decrease from age-25 BMD. Males have experienced an 18% decrease. It is important to note that this reduction in BMD does not necessarily imply a reduction in bone strength. In fact, the increase in outer radius observed above can compensate for the loss in BMD.

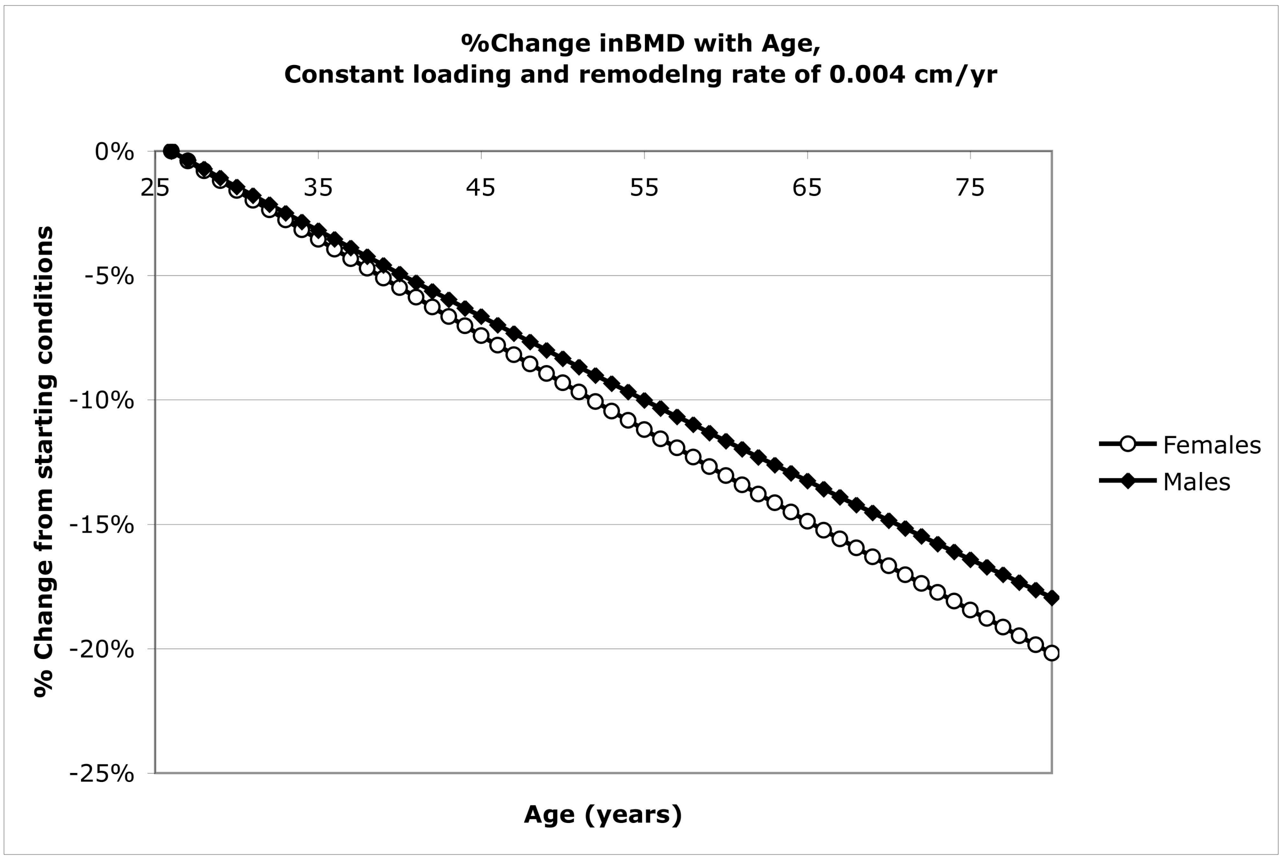

Figure 3: Percent change in BMD over time, for constant skeletal loading and 0.004 cm/yr remodeling

PROBLEM 2:

Effect of doubling and quadrupling the remodeling rate:

Doubling and quadrupling the remodeling rate are the equivalent of changing r_{rate} (rate of increase of the inner radius) to 0.008 cm/year and 0.016 cm/year, respectively. With a doubled remodeling rate, to maintain a constant section modulus, an 80-year-old female would have an outer femoral shaft radius of 1.62 cm (a 16% increase from age 25). An 80-year-old male would have an outer radius of 1.95 cm (a 15% increase from age 25). If the remodeling rate were quadrupled, up to 0.016 cm/year, an 80-year-old female would have an outer femoral shaft radius of 1.95 cm (a 39% increase from age 25) to maintain constant Z. An 80-year-old male experiencing this remodeling rate would have an outer radius of 2.28 cm (a 34% increase from age 25)! Combining these outer radius values with the age-80 inner radius values result in cortical thicknesses of only 0.17 cm (compared to 0.5 cm at age 25).

Figures 4 through 7 show the trends for percent changes in outer radius, inner radius, and BMD as time goes on, for the doubled and quadrupled remodeling rates.

Notice that it still appears to be physically possible to maintain bone strength even as 0.016 cm of the inner radius of the cortical shell is being absorbed each year.

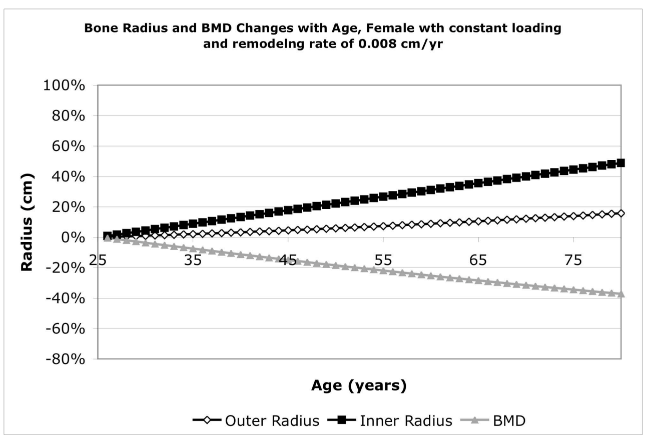

Figure 4: Female with 0.008 cm/year remodeling

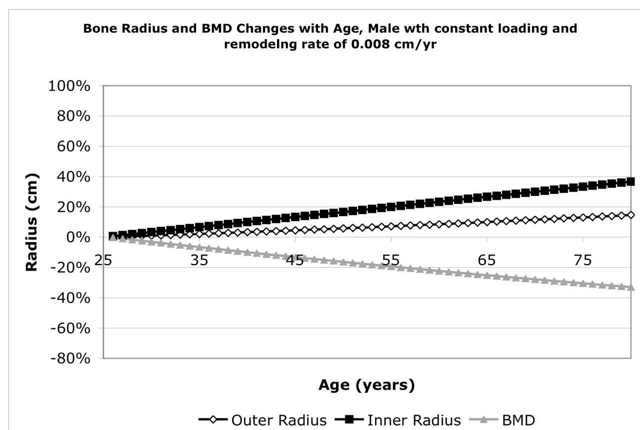

Figure 5: Male with 0.008 cm/year remodeling

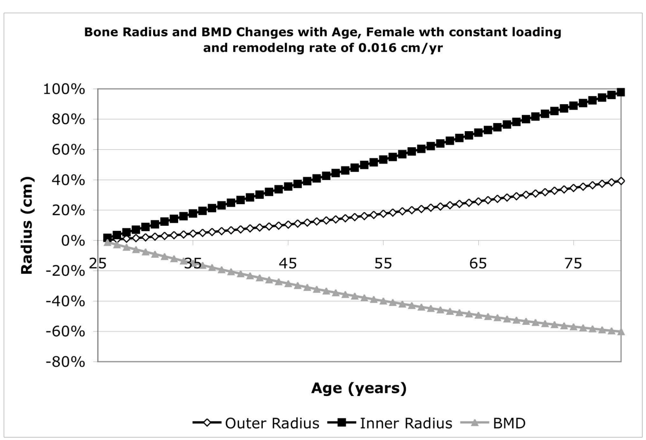

Figure 6: Female with 0.016 cm/year remodeling

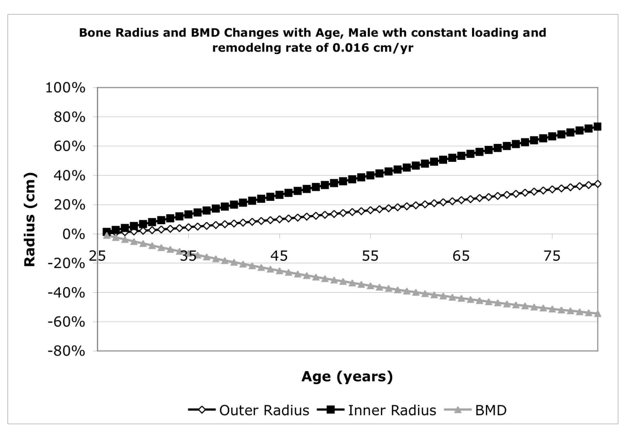

Figure 7: Male with 0.016 cm/year remodeling

PROBLEM 3.

Variable loading profile:
For skeletal loading to be reduced 30% by age 60, I assume that the loading profile decreases linearly between ages 35 and 60. Before age 35, it is constant at 100%, and after age 60, it is constant at 70%. In between these ages, I assume that a person's activity level gradually decreases and that consequently, his or her skeletal loading gradually decreases. The years between age 35 and 60 correspond to the time when a person's children might be adolescents and young adults, when they require much less physical activity and strain from their parents than they did when they were young children. Age 35 is also a typical time for people to start exercising less frequently and less strenuously and to begin losing muscle mass and muscle tone. Finally, age 35 is a typical time for a person to settle into a more sedentary job. Figure 8 depicts this loading profile that I have just explained; it is constant, gradually decreasing, and then again constant.

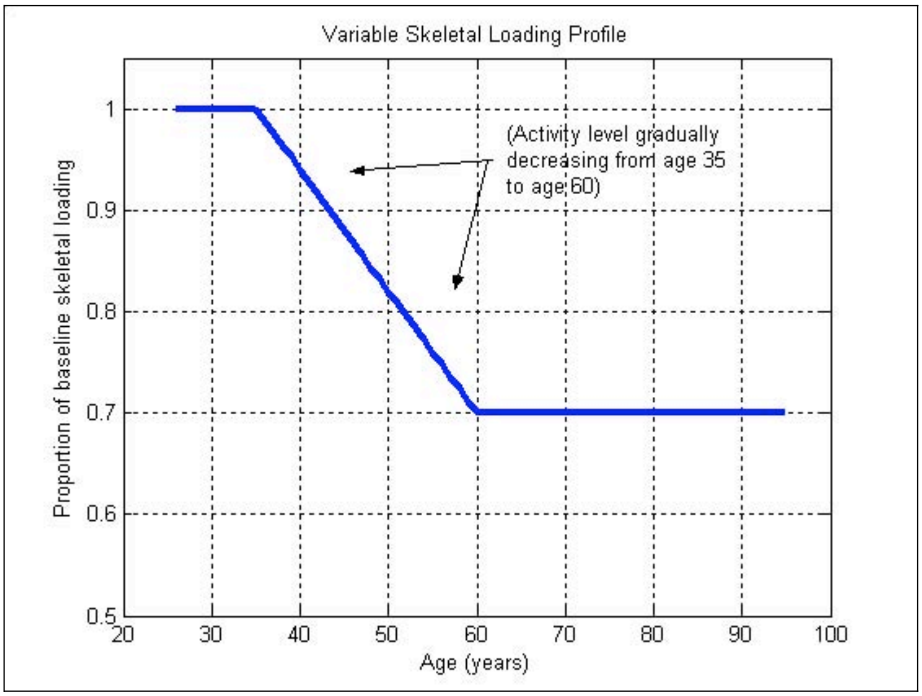

Figure 8: Variable skeletal loading profile resulting in 30% reduction by age 60

As stated in the problem, it is assumed that any reduction in skeletal loading is linearly related to a reduction in femoral shaft section modulus. Consequently, the gradual decrease from 100% to 70% loading corresponds to a gradual decrease from 100% to 70% of a person's initial section modulus. The picture of the percent change in section modulus over time is identical to the picture of skeletal loading over time.

This gradual reduction in skeletal loading has the effect of preventing the increase in the outer radius and consequently speeding up the decrease of BMD. The inner femoral radius continues to increase at the same 0.004 cm/year rate. Physically, this model predicts that remodeling continues to absorb bone from the inner cortical surface, but modeling ceases to add bone to the outer surface. The reduction in skeletal loading means that the bone has no reason to adapt to maintain a constant section modulus. Since the section modulus can be allowed to decrease, the outer radius does not have to be built upon. Figures 9 and 10 show how the outer radius remains almost constant while the inner radius increases linearly (as before) and the BMD decreases by as much as 40%. During the years from age 35 to 60, while the skeletal loading is decreasing gradually, the rate of change of BMD is faster than during the years while the skeletal loading is constant.

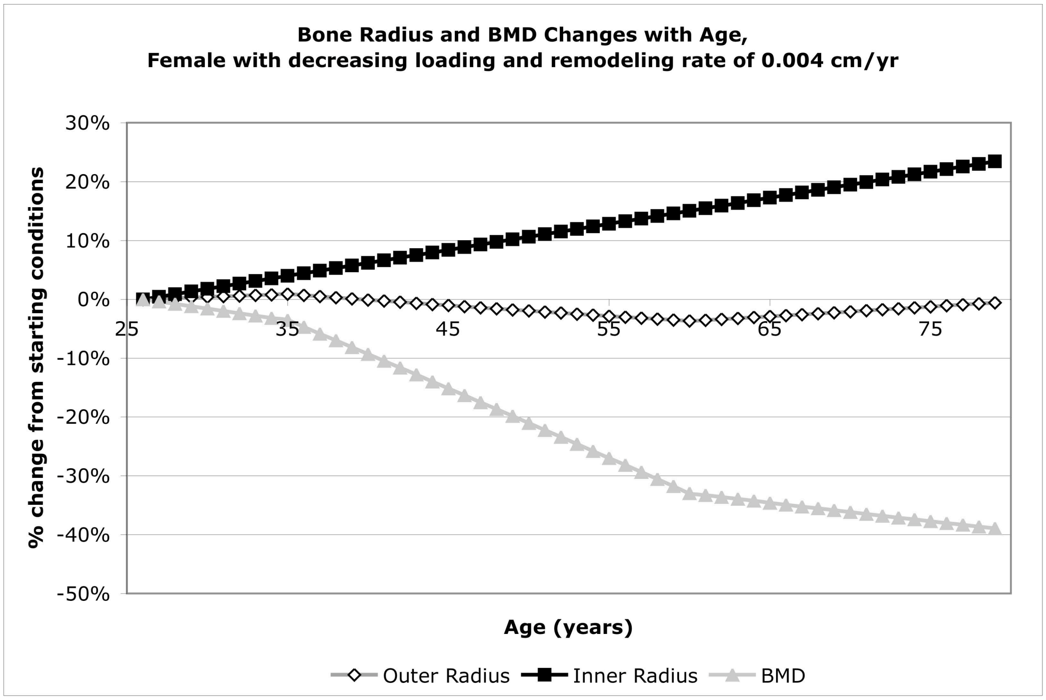

Figure 9: Percent changes in female BMD and cortical radii, subject to reductions in skeletal loading

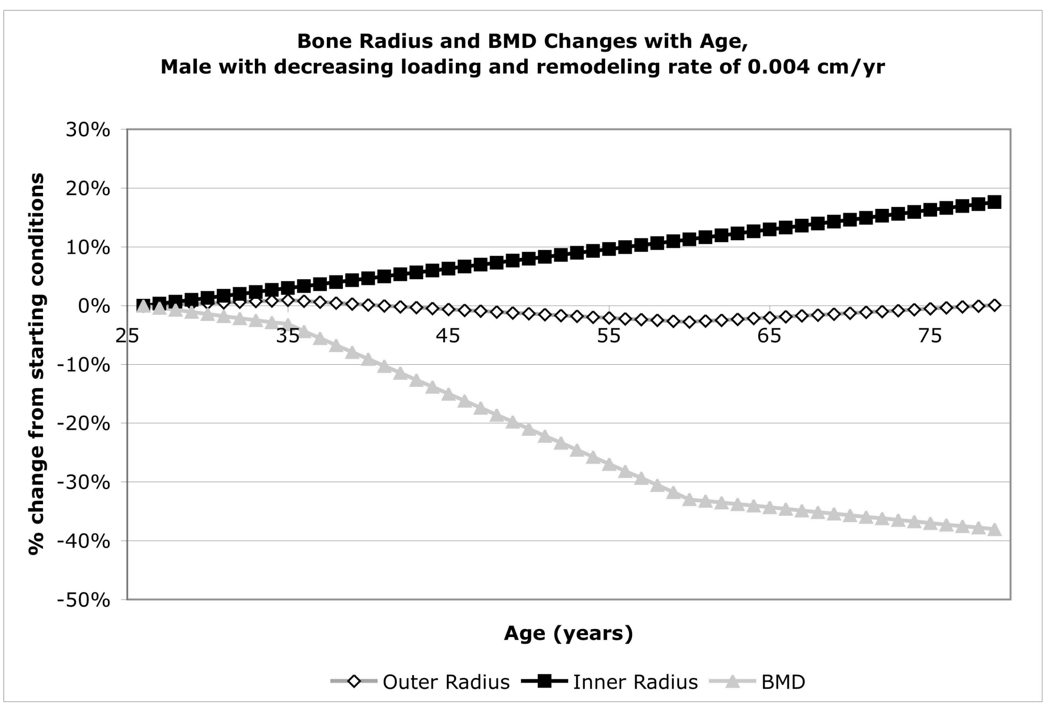

Figure 10: Percent changes in male BMD and cortical radii, subject to reductions in skeletal loading

Summary of loading and remodeling effects:

To understand better the profound effect on the femur caused by a reduction in skeletal loading , I created a summary table to compare the results of the different skeletal loading and bone remodeling scenarios.

Table 1: Summary of changes in outer radius, inner radius, and BMD, for constant and variable loading

Gender	Loading profile	Remodeling rate (cm/year)	Age 80 outer radius (cm)	Outer radius % change from 25 to 80 yrs	Age 80 inner radius (cm)	Inner radius % change from 25 to 80 years	Age 80 BMD	BMD % change from 25 to 80 years
Female	Constant	0.004	1.49	6.4	1.12	24	1.08	-20
	Constant	0.008	1.62	16	1.34	49	0.85	-37
	Constant	0.016	1.95	39	1.78	98	0.53	-61
	Linear 30% decrease between ages 35 and 60	0.004	1.39	-0.4	1.12	24	1.25	-39
Male	Constant	0.004	1.81	6.5	1.42	18	1.15	-18
	Constant	0.008	1.95	15	1.64	37	0.94	-33
	Constant	0.016	2.28	34	2.08	73	0.64	-54
	Linear 30% decrease between ages 35 and 60	0.004	1.71	0.3	1.42	18	0.87	-38

The summary table shows clearly that of the scenarios modeled, the largest values for percent change in BMD occur for a remodeling rate of 0.016 cm/year (the quadrupled rate). For this same remodeling rate, however, the values for percent change in outer radius are also at their largest. Consequently, bone strength is not necessarily reduced; losses in BMD are compensated for by gains in bone width.

However, the summary table also shows a scenario where bone strength likely is reduced. For the case when skeletal loading decreases with age, the percent change in BMD is about the same as in the doubled remodeling rate case, but there is no large increase in shaft outer radius to compensate for the loss in BMD. In the female case of decreasing skeletal loading, BMD decreases by 39% over the course of 55 years, and the outer radius actually decreases by 0.4%. This result of our simple model has serious implications for human spaceflight. When we model a load reduction of just 30%, we see the possibility for a severe reduction in bone strength. During spaceflight, however, loading is not simply reduced; rather, it is completely eliminated; there is zero skeletal loading in microgravity (with the exception of exercise countermeasures).

MATLAB Code

```
%Find outer radius as a cunction of changing inner radius
%assuming constant skeletal loading and consequently constant
%section modulus Z


ro_init = 1.70; %cm, for males

%ro_init = 1.40; %cm, for females

ri_init = 1.20; %cm, for males

%ri_init = 0.90; %cm, for females

ri_rate = 4*0.004; %cm/year

rho = 1.05; %g/cm^3, solid bone density


Z = 0.25*pi*(ro_init^4 - ri_init^4)/ro_init;     %section modulus


bone_results = zeros(70,7);         %matrix to hold results for each year

ro = ro_init;                       %initialize outer radius



for t = 1:70,

    bone_results(t,1) = t + 25;         %age in years

    ro_side = 0.25*pi*ro^4 - Z*ro;      %one side of equation for Z

    ri = ri_init + ri_rate*t;

    ri_side = 0.25*pi*ri^4;             %other side of equation for Z


    while (ro_side <= ri_side),

        ro = ro + 0.00005;              %make tiny increment in ro if one

        ro_side = 0.25*pi*ro^4 - Z*ro;  %side of equation is less than

    end                                 %other


    area = pi*(ro^2 - ri^2);            %once the two sides converge,

    BMD = area*rho/(2*ro);              %store results for this t

    bone_results(t,2) = ro;             %in matrix

    bone_results(t,3) = ri;

    bone_results(t,4) = ro_side;

    bone_results(t,5) = ri_side;

    bone_results(t,6) = area;

    bone_results(t,7) = BMD;


    t = t + 1;                          %move on to the next year


end
```

```
%Find outer radius as a cunction of changing inner radius
%assuming variable skeletal loading and consequently variable
%section modulus Z

%ro_init = 1.70; %cm, for males
ro_init = 1.40; %cm, for females
%ri_init = 1.20; %cm, for males
ri_init = 0.90; %cm, for females
ri_rate = 0.004; %cm/year
rho = 1.05; %g/cm^3, solid bone density

Z_init = 0.25*pi*(ro_init^4 - ri_init^4)/ro_init;     %section modulus
Z = Z_init;

bone_results = zeros(70,8);         %matrix to hold results for each year
ro = ro_init;                       %initialize outer radius

for t = 1:70,
    bone_results(t,1) = t + 25;         %age in years
    if t >= 10,                         %corresponds to age 35
        if t <= 35,                     %corresponds to age 60
            Z = Z_init*(-0.0120*(t-10)+1);
                            %decreased loading ---> decreased Z
                            %this expression for changing Z will cause it
                            %decrease by 30% over 25 years

        end
    end


    ro_side = 0.25*pi*ro^4 - Z*ro;      %one side of equation for Z
    ri = ri_init + ri_rate*t;
    ri_side = 0.25*pi*ri^4;             %other side of equation for Z

       while (ro_side <= ri_side),
            ro = ro + 0.00005;
            ro_side = 0.25*pi*ro^4 - Z*ro;
        end

if (ro_side >= ri_side+0.004),
        ro = ro - 0.02;              %make tiny increment in ro if one
        ro_side = 0.25*pi*ro^4 - Z*ro;  %side is less than other

        while (ro_side <= ri_side),
            ro = ro + 0.00005;
            ro_side = 0.25*pi*ro^4 - Z*ro;
        end
end

    area = pi*(ro^2 - ri^2);            %once the two sides converge,
    BMD = area*rho/(2*ro);              %store results for this t

    bone_results(t,2) = ro;             %in matrix
    bone_results(t,3) = ri;
    bone_results(t,4) = ro_side;
    bone_results(t,5) = ri_side;
    bone_results(t,6) = area;
    bone_results(t,7) = BMD;
    bone_results(t,8) = Z;


    t = t + 1;                          %move on to the next year


end
```