16.512, Rocket Propulsion
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Lecture 35-36: Impulsive and Low-Thrust Maneuvers in Space

See Lectures 3-4 of 16.522 (Space Propulsion) for coverage of Low Thrust Maneuvers and Re-positing within an orbit.

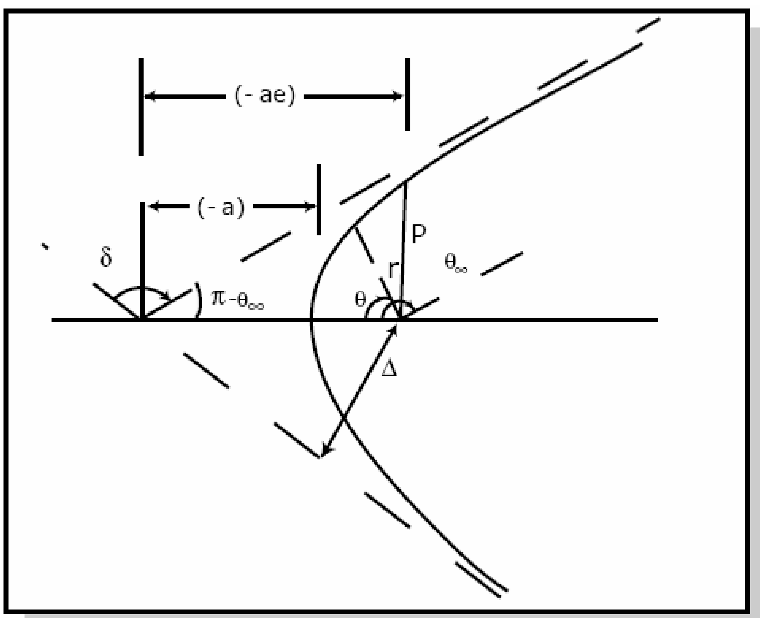
We add here material on Hyperbolic Orbits and Interplanetary Transfer.

Hyperbolic trajectories:

$$r = \frac{p}{1+e\cos\theta} \qquad e > 1$$

Asymptotes : $e\cos\theta = -1 \qquad \cos\theta = -\frac{1}{e} \qquad \theta = \boxed{\theta_\infty = \pi - \cos^{-1}\left(\frac{1}{e}\right)}$

$\boxed{h = \sqrt{\mu P}}$ still valid



Instead of "semimajor axis" a>0 distance from perigee to "center" is (-a), and we still have

$$\boxed{E = -\frac{\mu}{2a} > 0}$$

Also
$$p = a\left(1-e^2\right) = \left(-a\right)\left(e^2-1\right)$$

Similarly, distance |focus - center| is (-a)e (it is ae in ellipse)

Turning angle: $\delta = \pi - 2(\pi - \theta_\infty) = \pi - 2\cos^{-1}\left(\frac{1}{e}\right) = 2\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{e}\right)\right)$

$$\boxed{\delta = 2\sin^{-1}\left(\frac{1}{e}\right)}$$

Miss distance or Impact Parameter

$$\Delta = (-ae)\ \sin(\pi - \theta_\infty) = (-ae)\sin\theta_\infty \qquad \boxed{\Delta = (-ae)\sqrt{1-\frac{1}{e^2}}}$$

or $$\Delta = \frac{P}{e^2-1}\cancel{e}\sqrt{\frac{e^2-1}{\cancel{e^2}}} \qquad \boxed{\Delta = \frac{P}{\sqrt{e^2-1}}}$$

Excess hyperbolic velocity $$\boxed{V_\infty = \sqrt{2E} = \sqrt{\frac{\mu}{(-a)}} = \sqrt{\frac{\mu\left(e^2-1\right)}{p}}}$$

$\boxed{"C_3" = v_\infty^2}$ used in many books, reports

Orbit characterized by any pair of parameters, like
(p, e) , (v_p, θ_∞) , $(v_\infty, \theta_\infty)$, (v_∞, Δ) , (γ_p, v_∞) , (γ_p, δ)

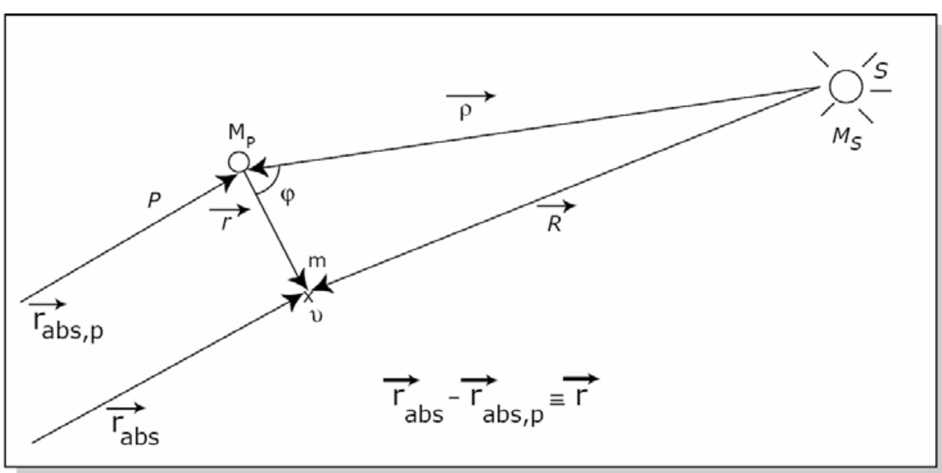
Example : Given (v_∞, Δ) , $-a = \dfrac{\mu}{v_\infty^2}$ $\quad \sqrt{e^2-1} = \dfrac{\Delta}{-a} = \dfrac{\Delta v_\infty^2}{\mu}$

then $$\delta = 2\sin^{-1}\frac{1}{e} = 2\cot^{-1}\sqrt{e^2-1} = 2\cot^{-1}\left(\frac{v_\infty^2\,\Delta}{\mu}\right)$$

Planetary "Spheres of Influence" (SOI)

Locus of points for which the ratio of (the Sun's disturbing acceleration of the relative vehicle-planet motion) to (the planet-induced vehicle acceleration) equals the ratio of (the planet-induced acceleration of the relative vehicle-Sun motion) (the Sun-induced vehicle acceleration).

This attracting centers, one vehicle



$$\cancel{m}\,\frac{d^2\,\overrightarrow{r_{abs}}}{dt^2} = -\frac{GM_P\,\cancel{m}}{r^3}\vec{r} - \frac{GM_s\,\cancel{m}}{R^3}\vec{R}$$

$$\cancel{M_P}\,\frac{d^2\,\overrightarrow{r_{abs,p}}}{dt^2} = -\frac{G\,\cancel{M_P}\,m}{r^3}\vec{r} - \frac{GM_s\,\cancel{M_P}}{\rho^3}\vec{\rho}$$

Subt.

$$\boxed{\underbrace{\frac{d^2\,\vec{r}}{dt^2}}_{\text{Accel. of vehicle Relative to planet}} = \underbrace{-\frac{G(M_P + m)}{r^3}\vec{r}}_{\text{Relative accel. due to planet only}} + \underbrace{GM_s\left(\frac{-\vec{R}}{R^2} + \frac{\vec{\rho}}{\rho^3}\right)}_{\text{Sun's effect on d. Notice: difference between attractions vehicle and on planet}}}$$

We could reverse the roles of planet and Sun, and get

$$\boxed{\frac{d^2\,\vec{R}}{dt^2} = -\frac{G(M_s + m)}{R^3}\vec{R} + GM_P\left(\frac{-\vec{\rho}}{\rho^3} - \frac{\vec{r}}{r^3}\right)}$$

So, *SOI* defined by locus of

$$\frac{M_s\left|\frac{\vec{\rho}}{\rho^3} - \frac{\vec{R}}{R^3}\right|}{\frac{M_p + m}{r^2}} = \frac{M_p\left|\frac{-\rho}{\rho^3} - \frac{\vec{r}}{r^3}\right|}{\frac{M_s + m}{R^2}}$$

Since $m << M_p << M_s$ and r<<p, R,

$$\frac{M_s}{M_p}\left|\frac{\vec{\rho}}{\rho^3} - \frac{\vec{R}}{R^3}\right| r^2 \simeq \frac{M_p}{M_s}\,\frac{1}{r^2}\,R^2$$

Also, $\vec{R} = \vec{\rho} + \vec{r}$ $\qquad \bar{R}^3 = (\rho^2 + \cancel{r^2} + 2\,\vec{\rho}\,.\,\vec{r})^{-3/2} \simeq \rho^{-3}\left(1 + \underbrace{\frac{2\,\vec{\rho}\,.\,\vec{r}}{\rho^2}}_{-\frac{r}{\rho}\cos\varphi}\right)^{-3/2} \simeq \rho^{-3}\left(1 - 3\,\frac{\vec{\rho}\,\vec{r}}{\rho^2}\right)$

$$\frac{\vec{\rho}}{\rho^3} - \frac{\vec{R}}{R^3} \simeq \frac{\vec{\rho}}{\rho^3} - \frac{1 - 3\,\frac{\vec{\rho}\,.\,\vec{r}}{\rho^2}}{\rho^3}\left(\vec{\rho} + \vec{r}\right) \simeq \frac{-\vec{r}}{\rho^3} + \frac{3\,\frac{\vec{\rho}\,.\,\vec{r}}{\rho^2}\,\vec{\rho}}{\rho^3} = \frac{r}{\rho^3}\left(-\vec{1}_r - 3\cos\varphi\,\vec{1}_\rho\right)$$

Magn.: $\frac{r}{\rho^3}$ $\qquad$ $\frac{3\,r\cos\varphi}{\rho^3}$

$$\left|\frac{\vec{\rho}}{\rho^3} - \frac{\vec{R}}{R^3}\right| = \frac{r}{\rho^3}\left(1 + 9\cos^2\varphi + 6\cos\varphi\overbrace{\left(1_r - 1_\rho\right)}^{-\cos\varphi}\right)^{1/2} \simeq \frac{r}{\rho^3}\sqrt{1 + 3\cos^2\varphi}$$

$$\frac{r^3}{\rho^3}\sqrt{1 + 3\cos^2\varphi} = \left(\frac{M_P}{M_s}\right)^2 \frac{R^2}{r^2}$$

$$\boxed{\frac{r}{R} = \frac{\left(M_p/M_s\right)^{2/5}}{\left(1 + 3\cos^2\varphi\right)^{1/10}}} \quad \rightarrow \quad \boxed{\frac{r}{R} \simeq \left(\frac{M_p}{M_s}\right)^{2/5}}$$

But $\left(1 + 3\cos^2\varphi\right)^{1/10}$ is between 1 and 1.15

Planet	SOI (Km)
Mercury	113,000
Venus	617,000
Earth	924,000
Jupiter	48.3×10^6
Neptune	86.7×10^6
Moon	66,000 (in E-M system) (17.2% of F_{EM}, not too small)

The Patched Conic Method for Interplanetary Transfers

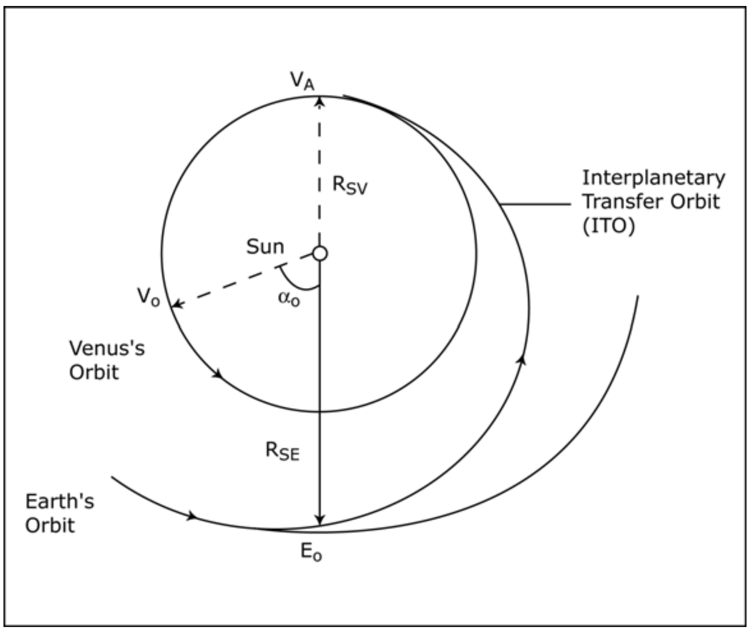
Since the SOI is typically small compared to the interplanetary distances, when dealing with trajectories that go from one planet to another we can make the approximation that only one body is attracting the space craft at each time. This is initially the first plane then the Sun, and finally the destination planet. Further, the "hand-over" from one planet to the sun or vice-versa can be assumed to be at the planet's location, when viewed on the Solar System scale. Care must be taken to converse momentum at these hand-over points.

Example: Hohmann transfer from Earth to Venus. Assume no plane changes are involved (in reality the Elliptical plane of the planetary orbits does not coincide with the equation plane of either, but we ignore this complication).

As a preliminary, notice that this type of (minimum Δv) maneuver requires a specific Earth–Venus configuration at launch; this configuration occurs once every Δt days, given by

$$\Delta t = \frac{1}{\left(\frac{1}{T_v}\right)-\left(\frac{1}{T_E}\right)} = \frac{1}{\frac{1}{224.7}-\frac{1}{365.3}} = 584 \text{ days} \tag{1}$$

The Earth head angle at launch (d_0 in the sketch) is calculated by stating that the time in the ITO (half the ellipse's period) is the same as that taken by Venus between V_o and V_i :



$$\frac{1}{2}\frac{2\pi}{\sqrt{\mu_s}}\left(\frac{R_{SE}+R_{SV}}{2}\right)^{3/2} = \left(\pi+\alpha_0\right)\frac{R_{SV}^{3/2}}{\sqrt{\mu_s}} \tag{2}$$

or

$$\alpha_0 = \pi\left[\left(\frac{R_{SE}+R_{SV}}{2R_{SV}}\right)^{3/2}-1\right] \tag{3}$$

Using R_{sv} = 0.7223 a.u , this gives $\alpha_0 = 54.0^0$.

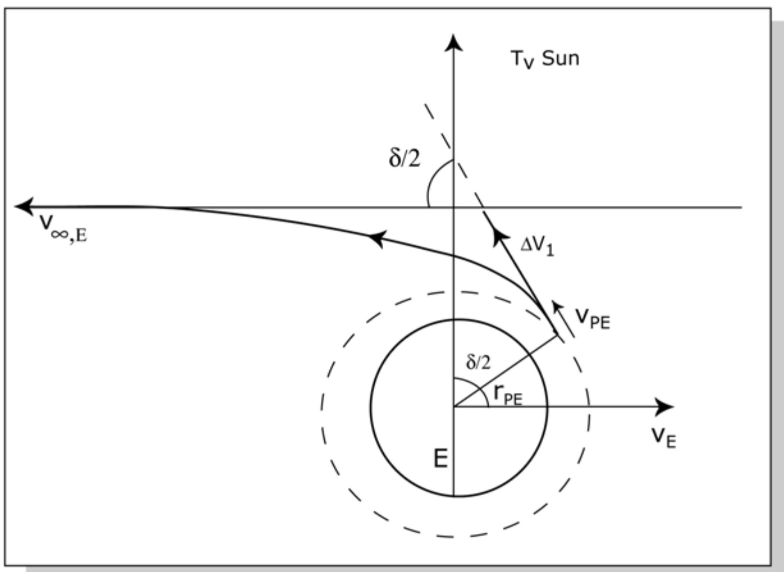
In the heliocentric part of the trajectory, the apohelion velocity (which the craft must have as it leaves Earth) is

$$v_a = \sqrt{\frac{\mu_s}{R_{SE}}\frac{2R_{SV}}{R_{SE}+R_{SV}}} \tag{4}$$

while the Earth's circular velocity is $v_E = \sqrt{\frac{\mu_s}{R_{SE}}}$. This is <u>more</u> than v_a, i.e., the spacecraft must leave the Earth's vicinity traveling <u>backwards</u> in the heliocentric frame. Later on it will overtake Earth as it accelerates in the transfer free-fall trajectory. The magnitude of this backwards velocity, which is the <u>hyperbolic excess velocity</u> when viewed from Earth, is

$$v_{\infty,E} = v_E - v_a = v_E\left(1 - \sqrt{\frac{2R_{SV}}{R_{SV} + R_{SE}}}\right) \tag{5}$$

The situation for time near launch, and when viewed in the Earth frame is as shown in the following sketch:



Conversation of energy <u>in the Earth's frame</u> gives

$$\frac{v_{\infty,E}^2}{2} = \frac{v_{PE}^2}{2} - \frac{\mu_E}{r_{PE}} \tag{6}$$

where υ_{PE} is the velocity after application of the escape firing at r_{PE}. Before this firing, the space craft was in orbit, at $v_{CE} = \sqrt{\frac{\mu E}{r_{PE}}}$, and so we find

$$v_{PE} = \sqrt{v_{\infty,E}^2 + \frac{2\mu_E}{r_{PE}}} \tag{7}$$

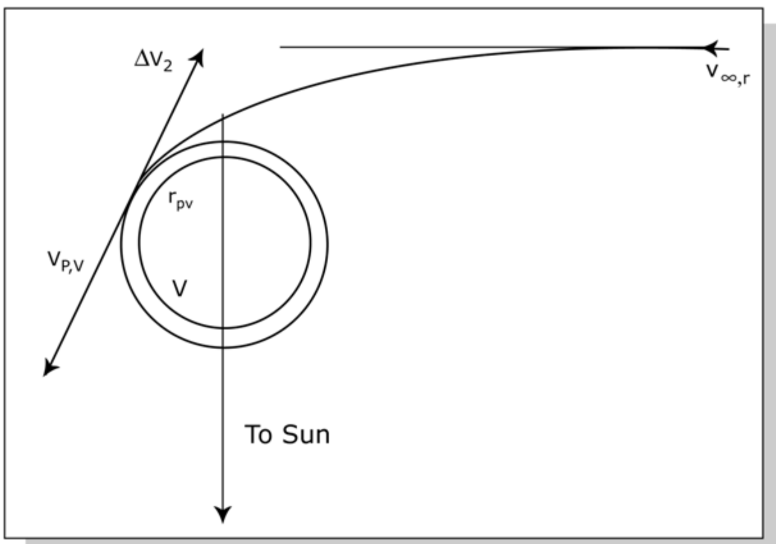
and therefore the single impulse needed to enter the ITO is

$$\Delta v_1 = v_{PE} - v_{CE} = \sqrt{\frac{\mu_E}{r_{PE}}} \left(\sqrt{2 + \frac{r_{PE}\, v_{\infty,E}^2}{\mu_E}} - 1 \right) \tag{8}$$

The point within LEO where this firing must occur is given, as shown, by $\delta/2$, where $\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$ is the total hyperbolic turning angle. Since $e = 1 + \frac{r_{P,E}}{(-a)} = 1 + \frac{r_{P,E}}{\mu_E}\, v_{\infty,E}^2$ we have all the elements to calculate this angle.

After traveling in the ITO, the spacecraft will approach Venus with an overtaking excess velocity

$$v_{\infty,v} = v_p - v_v = \sqrt{\frac{\mu_s}{R_{s,v}}} \left(\sqrt{\frac{2R_{SE}}{R_{SE} + R_{SV}}} - 1 \right) \tag{9}$$



and, working now in the Venus frame

$$\frac{v_{\infty.v}^2}{2} = \frac{v_{p.v}^2}{2} - \frac{\mu_v}{r_{\rho v}} \tag{10}$$

$$v_{c.v} = \sqrt{\frac{\mu_v}{r_{\rho v}}} \tag{11}$$

So that the circularization Δv is

$$\Delta v_2 = v_{\rho,v} - v_{c,v} = \sqrt{\frac{\mu_v}{r_{\rho v}}} \left(\sqrt{2 + \frac{r_{\rho v}\, v_{\infty,v}^2}{\mu_v}} - 1 \right) \tag{12}$$