Maxwellian Collisions

Maxwell realized early on that the particular type of collision in which the cross-section varies at $Q^*_{rs} \sim 1/g$ offers drastic simplifications. Interestingly, this behavior is physically correct for many charge-neutral collisions and moderate energies: The charge q polarizes the neutral in proportion to the field ($\alpha \sim q/r^2$) and the dipole α attracts the particle with a force $F \sim \alpha/r^3 \sim q/r^5$. From our work on power-law potentials, this is the interaction type that leads to $Q^* \sim 1/g$.

The simplification stems from the fact that the group $gQ^*_{rs}(g)$ appears in the integrals for $\vec{M}_{rs}$ and E_{rs}, and can now be moved outside as a constant. We put,

$$Q^*_{rs}(g) \;=\; \frac{g_0}{g} Q^*_{rs}(g_0)$$

then

$$\vec{M}_{rs} \;=\; \mu_{rs} g_0 Q^*_{rs}(g_0) \int\limits_{ws}\int\limits_{wr} f_r f_s (\vec{w}_r - \vec{w}_s) d^3w_r d^3w_s \;=$$

$$= \;\mu_{rs} g_0 Q^*_{rs}(g_0)\left[\underbrace{\int_{w_s} f_s d^3w_s}_{n_s} \underbrace{\int_{w_r} \vec{w}_r f_r d^3w_r}_{n_r\vec{u}_r} \;-\; \underbrace{\int_{w_r} f_r d^3w_r}_{n_r} \underbrace{\int_{w_s} \vec{w}_s f_s d^3w_s}_{n_s\vec{u}_s}\right]$$

$$\vec{M}_{rs} \;=\; \mu_{rs} g_0 Q^*_{rs}(g_0) n_r n_s (\vec{u}_r - \vec{u}_s)$$

Define ν_{sr} as the collision frequency of one s-particle will all r-particles,

$$\boxed{\nu_{sr} \;=\; n_r g Q^*_{rs}(g)} \quad \text{constant for Maxwellian collisions} \tag{1}$$

Similarly, $\nu_{rs} \;=\; n_s g Q^*_{rs}(g)$ (Note: $\nu_{sr}/n_r = \nu_{rs}/n_s$)

$$\boxed{\vec{M}_{rs} \;=\; \mu_{rs} n_s \nu_{sr} (\vec{u}_r - \vec{u}_s)} \;=\; \mu_{rs} n_r \nu_{rs} (\vec{u}_r - \vec{u}_s) \tag{2}$$

For other types of collisions the evaluation is much less straightforward, as it requires prior solution for f_r and f_s. However, the form $\vec{M}_{rs} \;=\; \mu_{rs} n_r \nu_{rs} (\vec{u}_r - \vec{u}_s)$ can always be recovered, only the collision frequency ν_{rs} is generally not a constant, but a function of the electron temperature, and is calculated from some of the existing models for f_r and f_s.

For energy transfer, we will deal directly with the internal energy transfer rate,

$$E'_{rs} = E_{rs} - \vec{u}_s \cdot \vec{M}_{rs} \tag{3}$$

From the definitions,

$$E'_{rs} = \mu_{rs} \int\limits_w \int\limits_{w_1} f_{r_1} f_s g Q^*_{rs}(g)(\vec{G} - \vec{u}_s) \cdot \vec{g} d^3w d^3w_1 \tag{4}$$

and for Maxwellian collisions, the group $gQ^*_{rs}(g)$ is a constant and moves outside the integration. The velocity combination inside can be manipulated next. Define the random velocities $\vec{c}_s = \vec{w} - \vec{u}_s,\ \vec{c}_r = \vec{w}_1 - \vec{u}_r$:

$$(\vec{G} - \vec{u}_s) \cdot \vec{g} = \tfrac{m_s(\vec{u}_s + \vec{c}_s) + m_r(\vec{u}_r + \vec{c}_r)}{m_r + m_s} \cdot (\vec{u}_r + \vec{c}_r - \vec{u}_s - \vec{c}_s) - \vec{u}_s \cdot (\vec{u}_r + \vec{c}_r - \vec{u}_s - \vec{c}_s)$$

$$= \underbrace{\left(\frac{m_s \vec{u}_s + m_r \vec{u}_r}{m_r + m_s} - \vec{u}_s \right)}_{\frac{m_r}{m_r+m_s}(\vec{u}_r - \vec{u}_s)} \cdot (\vec{u}_r - \vec{u}_s) + \tfrac{m_r}{m_r+m_s} c_r^2 - \tfrac{m_s}{m_r+m_s} c_s^2 + (\text{Terms linear in } \vec{c}_r \text{ or } \vec{c}_s)$$

Calling for short $m_r + m_s = m$, and ignoring the linear terms, because they integrate to zero (notice $\langle \vec{c}_s \rangle_s = 0,\ \langle \vec{c}_r \rangle_r = 0$),

$$(\vec{G} - \vec{u}_s) \cdot \vec{g} = \frac{m_r}{m} (\vec{u}_r - \vec{u}_s)^2 + \frac{m_r c_r^2 - m_s c_r^2}{m} + (\text{Linear terms})$$

Substitute into (4):

$$E'_{rs} = \frac{\mu_{rs}}{m} (g Q^*_{rs}) \Big[m_r (\vec{u}_r - \vec{u}_s)^2 \int\limits_w \int\limits_{w_1} f_{r_1} f_s d^3w_1 d^3w + \dots$$
$$\dots + \int\limits_w \int\limits_{w_1} m_r c_r^2 f_{r_1} f_s d^3w_1 d^3w - \int\limits_w \int\limits_{w_1} m_s c_s^2 f_{r_1} f_s d^3w_1 d^3w \Big] \tag{5}$$

The first of the integrals is simply $n_r n_s$. The second can be reorganized into $\int\limits_{w} d^3 w f_s \int\limits_{w_1} f_{r_1} m_r c_r^2 d^3 w_1$, of which the inner integral yields $3kT_r^{'} n_r$, while the outer one gives n_s. With a similar argument for the third integral, we obtain

$$\boxed{E_{rs}^{'} = \frac{\mu_{rs}}{m} n_r n_s (g Q_{rs}^*)[m_r(\vec{u}_r - \vec{u}_s)^2 + 3k(T_r^{'} - T_s^{'})]} \tag{6}$$

This has an interesting structure. The $m_r(\vec{u}_r - \vec{u}_s)^2$ term represents an irreversible internal energy addition (heat) to species s from collisions with r, provided the two species drift at different mean velocities. The second term, in $(T_r^{'} - T_s^{'})$ is the transfer of heat from r to s when the two species have different temperatures. It is reversible, depending on the sign of $T_r^{'} - T_s^{'}$.

For completeness, we can now calculate the transfer of full kinetic energy, $E_{rs} = E_{rs}^{'} + \vec{u}_s \cdot \vec{M}_{rs}$, with the result

$$E_{rs} = \mu_{rs} n_r n_s (g Q_{rs}^*) \left[\frac{m_r \vec{u}_r + m_s \vec{u}_s}{m} \cdot (\vec{u}_r - \vec{u}_s) + \frac{3k}{m}(T_r^{'} - T_s^{'}) \right] \tag{7}$$

Some simple applications of the Momentum Equations

Electrons Ohm's Law - Except for high-frequency effects (of the order of the Plasma Frequency) or for very strong gradients (like in double layers), the inertia of electrons can be neglected in their momentum balance. Assume collisions of electrons happen with one species of ions and one of neutrals only:

$$0 + \nabla P_e^{'} = -e n_e(\vec{E} + \vec{u}_e \times \vec{B}) + n_e m_e[\nu_{ei}(\vec{u}_i - \vec{u}_e) + \nu_{en}(\vec{u}_n - \vec{u}_e)] \tag{8}$$

where we used $\mu_{ei} \cong m_e$, $\mu_{en} \cong m_e$. In many cases, $u_i \ll u_e$, $u_n \ll u_e$, and we can simplify the equation by introducing the electron current density,

$$\vec{j}_e = -e n_e \vec{u}_e \tag{9}$$

$$e n_e \vec{E} + \nabla P_e^{'} = \vec{j}_e \times \vec{B} + \frac{m_e}{e}(\nu_{ei} + \nu_{en}) \vec{j}_e$$

Divide by $\frac{m_e}{e}(\nu_{ei} + \nu_{en})$ and define,

$$\sigma = \frac{e^2 n_e}{m_e(\nu_{ei} + \nu_{en})} \qquad \text{(Electrical conductivity)} \tag{10}$$

$$\beta_e = \frac{eB}{m_e(\nu_{ei} + \nu_{en})} \qquad \text{(Hall parameter)} \tag{11}$$

so that $$\boxed{\sigma\left(\vec{E}+\frac{\nabla P_e'}{en_e}\right) \;=\; \vec{j}_e+\vec{j}_e\times\vec{\beta}_e} \tag{12}$$

<u>Notice</u>:

(a) Electron pressure gradients can drive electron current. This is sometimes called a "diamagnetic current".

(b) As a limit, if boundary conditions forbid currents, $j_e = 0$, then $\vec{E}+\frac{\nabla P_e'}{en_e}=0$, $\vec{E}=-\frac{\nabla P_e'}{en_e}$, which means density gradients can set up a field – the Ambipolar field. If $T_e' \cong const.$

$$-\nabla\phi \;=\; -\frac{kT_e'}{e}\frac{\nabla n_e}{n_e} \rightarrow \boxed{\phi \;=\; \phi_0+\frac{kT_e'}{e}\ln\frac{n_e}{n_{e_0}}} \tag{13}$$

Which strongly resembles the kinetic Boltzmann relationship (except this time we only look at averages).

(c) The Hall parameter is the ratio $\beta=\frac{\omega_{ce}}{\nu_e}$ of election gyro frequency to electron collision frequency. It can be large in low-density plasmas, even with moderate B fields.

(d) The current is <u>not</u> aligned with the driving fields. Additional deviations from the electric field result from $\vec{E}^*=\vec{E}+\frac{\nabla P_e'}{en_e}$

(e) Eq. (12) can be solved for $\vec{j}_e$ in terms of $\vec{E}^*=\vec{E}+\frac{\nabla P_e'}{en_e}$. Start by multiplying (cross products) times $\vec{\beta}_e$:

$$\sigma\vec{\beta}_e\times\vec{E}^* = \;\vec{\beta}_e\times\vec{j}_e+\underbrace{\vec{\beta}_e\times(\vec{j}_e\times\vec{\beta}_e)}_{\beta_e^2\vec{j}_e-\vec{\beta}_e(\beta_e\cdot\vec{j}_e)}$$

consider only the current perpendicular to $\vec{B}$, so that $\vec{B}_e\cdot\vec{j}_e=0$:

$$\vec{j}_e\times\vec{\beta}_e \;=\; \vec{\beta}_e^2\vec{j}_{e\perp}-\sigma\vec{\beta}_e\times\vec{E}^*$$

and substitute this into (12): $\sigma\vec{E}^*=\vec{j}_{e\perp}+\beta_e^2\vec{j}_{e\perp}-\sigma\vec{\beta}_e\times\vec{E}^*$, or

$$\boxed{\vec{j}_{e\perp} \;=\; \frac{\sigma}{1+\beta_e^2}\left(\vec{E}_\perp^*+\vec{\beta}_e\times\vec{E}_\perp^*\right)} \quad \text{plus} \quad \vec{j}_{e\parallel} \;=\; \sigma\vec{E}_\parallel^* \tag{14}$$

This is sometimes organized as a tensor equation. With z taken along $\vec{B}$:

$$\begin{Bmatrix} j_{ex} \\ \\ j_{ey} \\ \\ j_{ez} \end{Bmatrix} = \sigma \begin{pmatrix} \frac{1}{1+\beta_e^2} & \frac{\beta_e}{1+\beta_e^2} & 0 \\ -\frac{\beta_e}{1+\beta_e^2} & \frac{1}{1+\beta_e^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} E_x^* \\ \\ E_y^* \\ \\ E_z^* \end{Bmatrix} \tag{15}$$

which makes the anisotropy of the situation more clear. In Ionospheric Physics, this is also put as a conductivity tensor

$$\vec{j}_e = \vec{\vec{\sigma}}\vec{E}^* \cdot \qquad \vec{\vec{\sigma}} = \begin{pmatrix} \sigma_P & \sigma_H & 0 \\ -\sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{16}$$

$\sigma_P =$ "Pedersen conductivity" (very small in the ionosphere, $\beta_e \gg 1$)
$\sigma_H =$ "Hall conductivity" (intermediate)
$\sigma_\parallel = \sigma$ "Parallel conductivity" (very large in the ionosphere)

Ambipolar Diffusion

Consider a simple case with $B = 0$, negligible inertia. Write both, electron and ion momentum equations:

$$\begin{aligned} m_i n_e \frac{D\vec{u}_i}{Dt} + \nabla P_i' &= e\vec{E}n_e + n_e[m_e \nu_{ie}(\vec{u}_e - \vec{u}_i) + \mu_{in}\nu_{in}(\vec{u}_n - \vec{u}_i)] \\ \nabla P_e' &= -e\vec{E}n_e + m_e n_e[\nu_{ei}(\vec{u}_i - \vec{u}_e) + \nu_{en}(\vec{u}_n - \vec{u}_e)] \end{aligned}$$

Add together, note $n_e \nu_{ei} = n_i \nu_{ie}$ (and also $n_e = n_i$),

$$m_i n_e \frac{D\vec{u}_i}{Dt} + \nabla(P_i' + P_e') = n_{ei}\mu_{in}\nu_{in}(\vec{u}_n - \vec{u}_i) + \underbrace{m_e n_e \nu_{en}(\vec{u}_n - \vec{u}_e)}_{\text{usually smaller } \sim \frac{m_e}{m_i} \text{ or } \sqrt{\frac{m_e}{m_i}}}$$

Also, normally $\nabla T_e'/T_e' \ll \nabla n_e/n_e$. In addition let us assume that ion inertia can be also neglected in comparison with the other terms in the momentum balance (although keeping the term would be more general),

$$k(T_e' + T_i')\nabla n_e = -\, n_e \mu_{in}\nu_{in}(\vec{u}_i - \vec{u}_n)$$

or,

$$n_e(\vec{u}_i - \vec{u}_n) = -\frac{k(T_e' + T_i')}{\mu_{in}\nu_{in}}\nabla n_e$$

Sometimes neutrals return from recombination of ions, so,

$$n_e\vec{u}_i = -n_n\vec{u}_n \qquad \text{then} \qquad n_e(\vec{u}_i - \vec{u}_n) = \left(\vec{u}_i + \frac{n_e}{n_n}\vec{u}_i\right)n_e = (n_n + n_i)\frac{n_e}{n_n}\vec{u}_i$$

$$n_e\vec{u}_i \ = \ -\frac{n_n}{n_i + n_n}\frac{k(T_e^{'} + T_i^{'})}{\mu_{in}\nu_{in}}\nabla n_e \qquad \nu_{in} \ = \ n_n g_{in} Q_{in} \qquad n_i + n_n \ = \ \frac{\rho}{m_i} \ ; \ \mu_{in} = \frac{m_i}{2}$$

$$n_e\vec{u}_i \ = \ -\frac{\cancel{n_n}k(T_e^{'} + T_i^{'})}{\frac{\cancel{m_i}}{2}\frac{\rho}{\cancel{m_i}}\cancel{n_n}Q_{in}g_{in}}\nabla n_e$$

$$\boxed{n_e\vec{u}_i \ = \ -D_a\nabla n_e}$$

$$\boxed{D_a \ = \ \frac{2k(T_e^{i} + T_i^{'})}{\rho Q_{in} g_{in}}} \qquad \text{Ambipolar diffusivity}$$

Back to the electron equation, if we neglect <u>both</u> collision forces,

$$\nabla P_e^{'} \cong -e\vec{E}n_e \qquad kT_e^{'}\frac{\nabla n_e}{n_e} \cong e\nabla\phi$$

$$\nabla(e\phi - kT_e^{'}\ln\ n_e) \ = \ 0$$

$$\boxed{\phi - \phi_0 \ = \ \frac{kT_e^{'}}{e}\ln\frac{n_e}{n_{e_0}}} \qquad \text{Equivalent Boltzmann equilibrium}$$

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