## Kinetic Theory

Distribution function:

$$f(\vec{w},\vec{x})d^3wd^3x \equiv \text{ \# of particles in the element of phase space volume } (\vec{x},d^3x),(\vec{w},d^3w)$$

Knowledge of $f$ completely specifies the gas state.

Total number density:

$$n = \iiint f d^3w$$

Average of a quantity $\phi(\vec{w})$:

$$\bar{\phi} = \langle \phi \rangle = \frac{1}{n}\iiint \phi f d^3w$$

The mean velocity:

$$\langle \vec{w} \rangle = \vec{u} = \frac{1}{n}\iiint \vec{w} f d^3w$$

Random velocity:

$$\vec{c} = \vec{w} - \vec{u} = \vec{w} - \langle \vec{w} \rangle$$

note that $\langle \vec{c} \rangle = \langle \vec{w} - \langle \vec{w} \rangle \rangle = \langle \vec{w} \rangle - \langle \vec{w} \rangle = 0$.

Mean kinetic energy per particle:

$$\langle K \rangle = \frac{1}{n}\iiint \frac{1}{2}mw^2 f d^3w$$

$$\langle K \rangle = \frac{1}{2}m\left\langle (\vec{u}+\vec{c})^2 \right\rangle = \frac{1}{2}m\left\langle u^2 + c^2 + 2\vec{u}\cdot\vec{c} \right\rangle = \frac{1}{2}mu^2 + \frac{1}{2}m\left\langle c^2 \right\rangle$$

Before proceeding further, we define temperature as the thermal energy equivalent to the second term in the RHS of last equation, therefore,

$$\frac{3}{2}kT = \frac{1}{2}m\left\langle c^2 \right\rangle$$

for any distribution (could be non-equilibrium). In particular, for equilibrium, we use the Maxwellian distribution:

$$\frac{1}{2}m\left\langle c^2 \right\rangle = \frac{1}{2}m\frac{1}{n}\iiint c^2 f d^3w = \frac{1}{2}m\int_0^\infty c^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} 4\pi c^2 dc = \frac{3}{2}kT$$

where integration by parts was used to obtain the result.

Following with properties of the random velocity, the random mass flux along $i$ is $mnc_i$, while the random momentum flux along $j$ due to random motion along $i$ is $mnc_ic_j$.

For all possible values of $\vec{c}$ we have:

$$mn\left\langle c_ic_j \right\rangle \equiv \text{momentum flux associated with random particle motions}$$

This is the definition of the pressure tensor $P_{ij}$,

$$\vec{\vec{P}} = nm \langle \vec{c}\,\vec{c} \rangle = \begin{bmatrix} \langle c_x^2 \rangle & \langle c_x c_y \rangle & \langle c_x c_z \rangle \\ \langle c_x c_y \rangle & \langle c_y^2 \rangle & \langle c_y c_z \rangle \\ \langle c_x c_z \rangle & \langle c_y c_z \rangle & \langle c_z^2 \rangle \end{bmatrix} \text{ and the trace } Tr\vec{\vec{P}} = nm \left\langle c^2 \right\rangle = 3nkT$$

If $f(\vec{c}) = f(c)$ (isotropic distribution), then $Tr\vec{\vec{P}} = 3P$, where,

$$P = nm \left\langle c_x^2 \right\rangle = \left\langle c_x^2 \right\rangle = \left\langle c_x^2 \right\rangle$$

hence $P = nkT$ for any isotropic distribution.

For the non-diagonal elements ($i \neq j$) of the pressure tensor, $P_{ij} = -\tau_{ij}$. These are the shear stresses. And once more, if the distribution is isotropic (for $i = x$ and $j = y$),

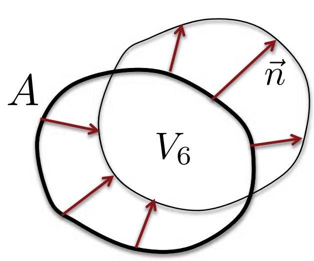
$$P_{xy} = -\tau_{xy} = m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_x c_y f(c) dc_x dc_y dc_z = 0$$

since the integral is over odd functions in both $c_x$ and $c_y$. If there are no shear stresses present, then there is no viscosity.

**Vlasov's Equation**

It should be possible, at least in principle, to compute the distribution function of particles in phase space given information about the forces acting on them. In particular, if we do not consider collisions, we can use Liouvilles theorem to find an equation for the distribution.

Consider the 6-dimensional phase space with $\vec{x}, \vec{w}$ as coordinates, and take a group of points initially occupying a volume $d^3x d^3w$ in it (namely, particles which are grouped about $\vec{x}$ in a volume $d^3x$, and of those only the ones with velocities in $d^3w$ about $\vec{w}$).




We take the volume large enough to contain many particles, and follow it for a certain time $dt$, always enclosing the <u>same</u> particles as they move in phase space. The rate of change of volume $V_6$ in this (or any) space is,

$$\frac{dV_6}{dt} = \oint \vec{v} \cdot \vec{n} dA$$

where $\vec{v}$ is the velocity vector of points in the area element $dA$ with normal $\vec{n}$. This vector is also 6-dimensional,

$$\vec{v} = \left[ \frac{d\vec{x}}{dt}, \frac{d\vec{w}}{dt} \right]$$

Using Gauss' theorem,

$$\frac{dV_6}{dt} = \oint \vec{v} \cdot \vec{n} dA = \int_{V_6} \nabla_6 \cdot \vec{v} dV_6 \quad \text{where} \quad \nabla_6 \cdot \vec{v} = \frac{\partial}{\partial x_i} \left( \frac{dx_i}{dt} \right) + \frac{\partial}{\partial w_i} \left( \frac{dw_i}{dt} \right)$$

where the convention of summation over repeated indices is implied. In this expression we note that $dx_i/dt = w_i$, and since the partial derivative $\partial/\partial x_i$ is evaluated at $w_j \equiv$ constant for any $j$, the first summation is identically zero. In the second summation, $dw_i/dt = a_i = F_i/m$ is the particle acceleration, which depends on the forces acting on the particle. Under some conditions on the forces, this term could also be zero:

1. If the force is conservative (can be derived from a potential), then $\vec{F} = \vec{F}(\vec{x})$ and,

$$\left(\frac{\partial \vec{F}}{\partial \vec{w}}\right)_{\vec{x}} \equiv 0$$

2. If there are, in addition to conservative forces, only magnetic forces coming from an external $\vec{B}(\vec{x})$. In that case,

$$F_i = q\left(\vec{w} \times \vec{B}\right)_i = q\varepsilon_{ijk} w_j B_k \quad \text{and} \quad \frac{\partial F_i}{\partial w_i} = q\varepsilon_{ijk}\frac{\partial w_j}{\partial w_i} B_k = q\varepsilon_{ijk}\delta_{ij}B_k = 0$$

   since $\delta_{ij} = 1$ only for $i = j$, but $\varepsilon_{ijk} = 0$ for any index repetition.

These two conditions cover the case of a collisionless plasma in external $\vec{E}$ and $\vec{B}$ fields, in such cases $dV_6/dt = 0$ and since we follow the same particles, it follows that the phase space density (the distribution function $f$) is constant along particle trajectories. Noting that,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{w} \cdot \nabla_x f + \vec{a} \cdot \nabla_w f$$

we obtain Vlasov's equation governing $f(\vec{x}, \vec{w})$ in collisionless situations for a species $s$,

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + w_i \frac{\partial f_s}{\partial x_i} + \frac{F_i}{m_s}\frac{\partial f_s}{\partial w_i} = 0 \qquad \text{where} \qquad F_i = q_s\left(\vec{E} + \vec{w} \times \vec{B}\right)_i$$

If $\vec{E}$ and $\vec{B}$ are determined in a self-consistent way, as a result of the collective effect of all the plasma particles, this Vlasov equation can still be used as an approximation in finding the distribution function, $f$.

### The Boltzmann Equation

Interactions between particles (like collisions) are excluded in the preceding formulation, because if they exist, the last three components of $\vec{v}$ (namely, the acceleration) are no longer functions of position in phase space alone $(\vec{x}, \vec{w})$, but also of the position of the other particles and then Gauss' theorem does not apply. It is still true that,

$$\frac{dV_6}{dt} = \oint \vec{v} \cdot \vec{n} dA$$

but not equal to,

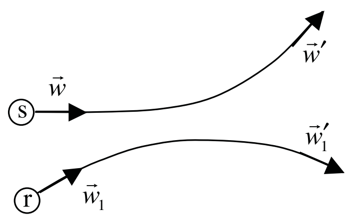
$$\int_{V_6} \nabla_6 \cdot \vec{v} dV_6$$

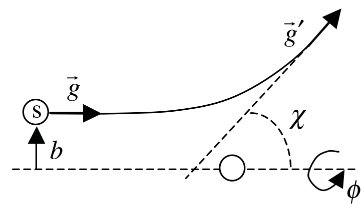
If we are willing to concentrate on binary collisions and on evolution times long compared to the intercollision time, then the rate of change of $f_s$ can be calculated separately, thus giving a RHS side to Vlasov's equation.

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + w_i \frac{\partial f_s}{\partial x_i} + \frac{F_i}{m_s}\frac{\partial f_s}{\partial w_i} = \left(\frac{df_s}{dt}\right)_{\text{coll}} \qquad \text{Boltzmann Equation}$$

Earlier we studied binary collisions in the relative frame. The velocities of the target and colliding particles can be written in terms of the center of mass and relative velocities, respectively:

$$\vec{w} = \vec{G} + \frac{m_r}{m_r + m_s}\vec{g} \quad \text{and} \quad \vec{w}_1 = \vec{G} - \frac{m_s}{m_r + m_s}\vec{g}$$




The relative velocity after the collision $\vec{g}'$ is a rotation of $\vec{g}$ through $(\chi, \phi)$, so given those angles (or the impact parameter $b$ and $\phi$) the resultant velocities in the laboratory frame $\vec{w}'$ and $\vec{w}_1'$ are just linear functions of $\vec{w}$ and $\vec{w}_1$. The Jacobian of the transformation,

$$\frac{\partial(\vec{w}', \vec{w}_1')}{\partial(\vec{w}, \vec{w}_1)} \qquad \text{is unity.}$$

To see this, decompose into three Jacobians,

$$\frac{\partial(\vec{w}', \vec{w}_1')}{\partial(\vec{w}, \vec{w}_1)} = \frac{\partial(\vec{w}', \vec{w}_1')}{\partial(\vec{G}', \vec{g}')} \frac{\partial(\vec{G}', \vec{g}')}{\partial(\vec{G}, \vec{g})} \frac{\partial(\vec{G}, \vec{g})}{\partial(\vec{w}, \vec{w}_1)}$$

For the second term, note that $\vec{G} = \vec{G}'$,

$$\frac{\partial(\vec{G}', \vec{g}')}{\partial(\vec{G}, \vec{g})} = \frac{\partial \vec{g}'}{\partial \vec{g}} = 1 \text{ , which is just a rotation.}$$

For the third term we write,

$$\frac{\partial(\vec{G}, \vec{g})}{\partial(\vec{w}, \vec{w}_1)} = \begin{vmatrix} \frac{\partial \vec{G}}{\partial \vec{w}} & \frac{\partial \vec{G}}{\partial \vec{w}_1} \\ \frac{\partial \vec{g}}{\partial \vec{w}} & \frac{\partial \vec{g}}{\partial \vec{w}_1} \end{vmatrix} = \begin{vmatrix} G_{x_{w_x}} & G_{x_{w_y}} & G_{x_{w_z}} & G_{x_{w_{1x}}} & G_{x_{w_{1y}}} & G_{x_{w_{1z}}} \\ G_{y_{w_x}} & G_{y_{w_y}} & G_{y_{w_z}} & G_{y_{w_{1x}}} & G_{y_{w_{1y}}} & G_{y_{w_{1z}}} \\ G_{z_{w_x}} & G_{z_{w_y}} & G_{z_{w_z}} & G_{z_{w_{1x}}} & G_{z_{w_{1y}}} & G_{z_{w_{1z}}} \\ g_{x_{w_x}} & g_{x_{w_y}} & g_{x_{w_z}} & g_{x_{w_{1x}}} & g_{x_{w_{1y}}} & g_{x_{w_{1z}}} \\ g_{y_{w_x}} & g_{y_{w_y}} & g_{y_{w_z}} & g_{y_{w_{1x}}} & g_{y_{w_{1y}}} & g_{y_{w_{1z}}} \\ g_{z_{w_x}} & g_{z_{w_y}} & g_{z_{w_z}} & g_{z_{w_{1x}}} & g_{z_{w_{1y}}} & g_{z_{w_{1z}}} \end{vmatrix} \qquad \text{where} \quad G_{x_{w_x}} = \frac{\partial G_x}{\partial w_x}$$

and using the transformation equations (with $m = m_s + m_r$),

$$\vec{G} = \frac{m_s\vec{w} + m_r\vec{w}_1}{m} \qquad \text{and} \qquad \vec{g} = \vec{w} - \vec{w}_1$$

we obtain,

$$\frac{\partial(\vec{G},\vec{g})}{\partial(\vec{w},\vec{w}_1)} = \begin{vmatrix} m_s/m & 0 & 0 & m_r/m & 0 & 0 \\ 0 & m_s/m & 0 & 0 & m_r/m & 0 \\ 0 & 0 & m_s/m & 0 & 0 & m_r/m \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{vmatrix} = 1$$

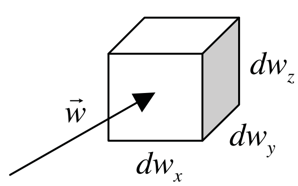
Similarly, for the first term,

$$\frac{\partial(\vec{w}',\vec{w}_1')}{\partial(\vec{G}',\vec{g}')} = \begin{vmatrix} \frac{\partial\vec{w}'}{\partial\vec{G}'} & \frac{\partial\vec{w}'}{\partial\vec{g}'} \\ \frac{\partial\vec{w}_1'}{\partial\vec{G}'} & \frac{\partial\vec{w}_1'}{\partial\vec{g}'} \end{vmatrix} = 1$$

and therefore,

$$\frac{\partial(\vec{w}',\vec{w}_1')}{\partial(\vec{w},\vec{w}_1)} = 1 \text{ , the Jacobian is unity.}$$

The effect of collisions on particles will be to remove or add them to the phase space volume considered in Boltzmann's equation.




Depopulation by collision depletion: Consider a mixture of species, and assume that every collision changes $\vec{w}$ of the target particle enough to instantaneously remove it from $d^3w$ (note that it is removed from the phase space volume even though it remains inside $d^3x$), while the colliding particle (which must have been originally outside $d^3w$) will remain outside as well. The problem then reduces to counting the number of collisions in that volume, per unit time,

$$\left(\frac{df_s}{dt}\right)_{\text{dep.}} d^3w$$

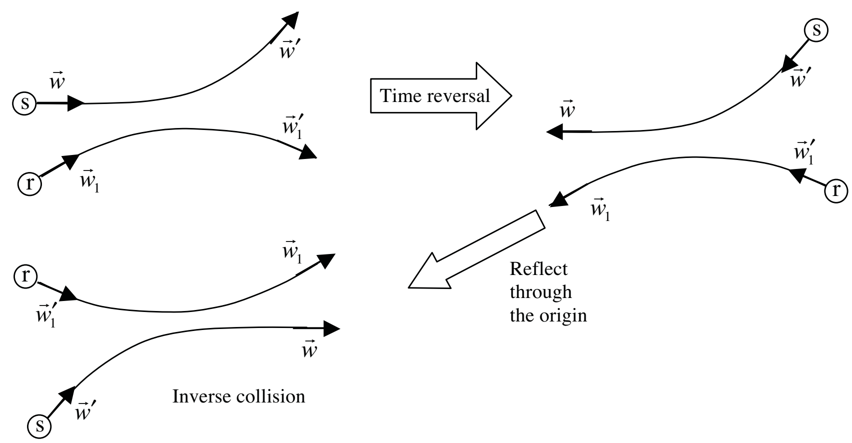
This number will be proportional to the number of particles $s$ contained in that volume $f_s(\vec{w})d^3w$, to the number of particles $r$ that will collide with them and the magnitude of the relative velocity, $g$. Given the differential cross section $\sigma_{rs}$, the number of depleting collisions per unit time, per unit volume is,

$$f_s(\vec{w})d^3w \int_{w_1}\int_{\Omega} f_r(\vec{w}_1)d^3w_1 g\sigma_{rs}d\Omega$$

therefore,

$$\left(\frac{df_s}{dt}\right)_{\text{dep.}} = -\int_{w_1}\int_{\Omega} f_s(\vec{w})f_r(\vec{w}_1)d^3w_1 g\sigma_{rs}d\Omega$$

Replenishing collisions: There are also collisions occurring (inside $d^3x$) in which one of the particles has a final velocity in $d^3w$. To count these, we specify each one by the final velocity $\vec{w}$ of one particle (which is prescribed), and the final velocity $\vec{w}_1$ of the other particle (which will be summed over). To do this, we take the inverse of the collision, by first reversing in time a "normal" collision and then reflecting through the origin:




In a similar way as before, we can count the number of collisions of this kind per unit time, per unit volume, which is,

$$f_s(\vec{w}')d^3w' \int_{w_1'}\int_{\Omega} f_r(\vec{w}_1')d^3w_1' g'\sigma_{rs}d\Omega$$

and since $g = g'$ and, as the Jacobian of the transformation is unity ($d^3w'd^3w_1' = d^3wd^3w_1$), we have,

$$\left(\frac{df_s}{dt}\right)_{\text{rep.}} = \int_{w_1}\int_{\Omega} f_s(\vec{w}')f_r(\vec{w}_1')d^3w_1 g\sigma_{rs}d\Omega$$

Adding both terms and counting all species $r$ (including $s$), we obtain Boltzmann's collision integral,

$$\left(\frac{df_s}{dt}\right)_{\text{coll}} = \sum_r \int_{w_1}\int_{\Omega} (f_s'f_{r_1}' - f_sf_{r_1})g\sigma_{rs}d^3w_1 d\Omega$$

where $f_s' = f_s(\vec{w}')$, $f_{r_1}' = f_r(\vec{w}_1')$, $f_s = f_s(\vec{w})$, $f_{r_1} = f_r(\vec{w}_1)$ and $\sigma_{rs} = \sigma_{rs}(g, \Omega)$. Note that Boltzmann equation becomes non-linear when $r = s$.

Recall limitations:

1. Only binary collisions considered.

2. Since $f(\vec{w})f(\vec{w}')$ is taken to be proportional to the probability of finding molecules at $\vec{w}$ and $\vec{w}_1$, the molecular motions are assumed uncorrelated (molecular chaos). Must fail near the critical point of a gas, and also, in principle, in highly ionized gases, where forces are collective.

3. No inelastic effects have been considered. In general, have to add a term,

$$\left(\frac{df_s}{dt}\right)_{\text{inelastic}}$$

to include processes such as ionization or excitation.