LECTURE # 12

RIGID BODY DYNAMICS

- IMPLICATIONS OF $\vec{M} = \dot{\vec{H}}^{I}$
- GENERAL ROTATIONAL DYNAMICS
  - EULER'S EQUATION OF MOTION
- TORQUE FREE SPECIAL CASES.

PRIMARY LESSONS:

- 3D ROTATIONAL MOTION MUCH MORE COMPLEX THAN PLANAR (2D)
- EULER'S E.O.M. PROVIDE STARTING POINT FOR ALL A/C + S/C DYNAMICS
- SOLUTIONS TO EULER'S EQUATIONS ARE COMPLEX, BUT WE CAN DEVELOP GOOD GEOMETRIC VISUALIZATION TOOLS.

- NOW CAN DEVELOP THE FULL SET OF ROTATIONAL DYNAMICS :

$$\vec{M} = \dot{\vec{H}}^{I} = \dot{\vec{H}}^{B} + \vec{\omega} \times \vec{H}$$

TRANSPORT THM

B: DENOTES BODY FRAME.

ANGULAR VELOCITY OF BODY WRT INERTIAL.

- NOW, WE ASSUME THAT WE ARE USING A FRAME FOR THE BODY THAT IS CENTERED AT THE CENTER OF MASS AND <u>FIXED</u> TO THE BODY.

$$\Rightarrow \quad \dot{\vec{\vec{I}}}^{B} \equiv 0 \qquad \text{INERTIA VALUES FIXED.}$$

$$\therefore \quad \dot{\vec{H}}^{B} \equiv \frac{d^{B}}{dt}\left(\vec{\vec{I}} \cdot \vec{\omega}\right) = \vec{\vec{I}} \cdot \dot{\vec{\omega}}^{B}$$

$\vec{i}, \vec{j}, \vec{K}$ UNIT VECTORS OF BODY FRAME

RECALL, IF $\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{K}$

THEN $\dot{\vec{\omega}}^{B} = \dot{\omega}_x \vec{i} + \dot{\omega}_y \vec{j} + \dot{\omega}_z \vec{K}$

- <u>SUMMARY</u> :

$$\vec{M} = \dot{\vec{H}}^{I} = \vec{\vec{I}} \cdot \dot{\vec{\omega}}^{B} + \vec{\omega} \times \left(\vec{\vec{I}} \cdot \vec{\omega}\right)$$

– GENERAL FORM OF ROTATIONAL DYNAMICS.

- IF WE NOW USE THE BODY FRAME, CAN WRITE THESE IN MATRIX FORM:

$$M_B = I_B \dot{\omega}_B + \omega_B^{\times} I_B \omega_B$$

$$I_B = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yz} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

- VERY COMPLEX FOR FULL I_B

⇒ SIMPLIFIES IF WE ASSUME THAT BODY FRAME ALIGNED WITH <u>PRINCIPAL</u> <u>AXES.</u>

$$\Rightarrow I_B = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

- REDUCES TO <u>EULER'S</u> <u>EQUATIONS OF MOTION</u>:

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{bmatrix}$$

- EULER'S EQUATIONS
  - NONLINEAR, COUPLED, FEW ANALYTIC SOLUTIONS.

- TYPICALLY TWO PROBLEMS OF INTEREST:
  1. GIVEN $\vec{M}$, WHAT IS THE RESPONSE OF THE SYSTEM (GIVEN A MOTION, WHAT MUST $\vec{M}$ BE?)
  2. IN THE ABSENCE OF $\vec{M}$ (TORQUE FREE) WHAT WOULD THE MOTION OF THE BODY BE?

1. "GIVEN MOTION, FIND $\vec{M}$" IS RELATIVELY SIMPLE.
   MUCH HARDER THE OTHER WAY (GIVEN $\vec{M}$, FIND $\vec{\omega}(t)$)
   - REQUIRES SOLUTION OF THE COUPLED, NONLINEAR EQUATIONS - FEW ANALYTIC ANSWERS.
     ↳ EASILY DONE NUMERICALLY

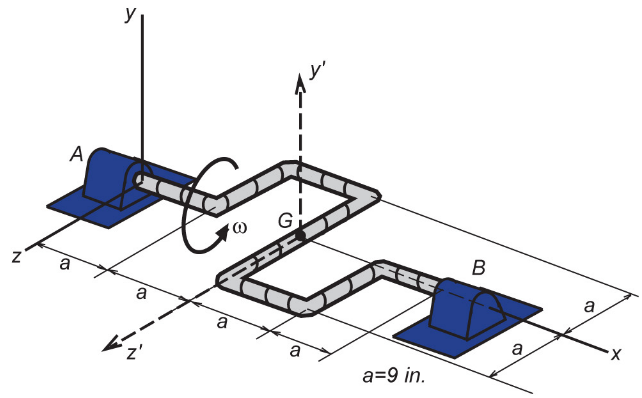
   ⇒ EXAMPLE

2. CAN GIVE A LOT OF GEOMETRIC INSIGHTS INTO WHAT TYPES OF MOTIONS OCCUR.
   - MOMENTUM + ENERGY ELLIPSOIDS.

   ⇒ "TORQUE FREE" MOTION ONLY.

EXAMPLE: BEER + JOHNSTON 18.67

- SHAFT WEIGHS 16-lb
- ROTATES AT CONSTANT RATE $\omega = 12$ RAD/SEC
- FIND REACTIONS AT POINTS A, B.



SOLUTION:

- FIX FRAME $xy'z'$ AT C.O.M. "G" WHICH ROTATES WITH THE FRAME.

• USE POINT G AND FRAME $xy'z'$, CAN CALCULATE THE INERTIAS:

$$I_x = \frac{10}{3} ma^2 \quad ; \quad I_{xy'} = 0 \quad ; \quad I_{xz'} = 2ma^2 \; ; \; \ldots$$

- CAN CALCULATE THE REST, BUT THIS IS ALL WE NEED, SINCE

$$H_G = I_G \omega_G = \begin{bmatrix} I_x & \cancelto{0}{I_{xy'}} & I_{xz'} \\ \cancelto{0}{I_{xy'}} & x & x \\ I_{xz'} & \boxed{x} & x \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_x \omega \\ 0 \\ I_{xz'} \omega \end{bmatrix}$$

∴ CAN EASILY SEE THAT H_G AND ω_G <u>ARE</u> <u>NOT</u> <u>ALIGNED</u> ⇒ TYPICAL OF 3D ROTATIONS BUT RARELY SEEN IN PLANAR PROBLEMS

- TO FIND REACTIONS, NEED TO FIND M_G

SET $M_G = \dot{H}_G^{I} = \cancel{\dot{H}_G^{G}}_{0} + \omega_G^{\times} H_G$

$$M_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} I_x \omega \\ 0 \\ I_{xz'} \omega \end{bmatrix} = \begin{bmatrix} 0 \\ -I_{xz'} \omega^2 \\ 0 \end{bmatrix}$$

- SO M_G IS <u>NON-ZERO</u> DUE TO THE NON-ZERO CROSS-MOMENT $I_{xz'}$
  - CAN THEN FIND THE REATIONS AT POINTS A, B TO APPLY THIS MOMENT ABOUT THE Y'-AXIS

$$\vec{F}_A = -\vec{F}_B = \frac{1}{2} M a \omega^2 \vec{K}$$ IN Z' DIRECTION

⇒ FORCE COUPLE IN Z' DIRECTION, WHICH ROTATES WITH THE FRAME.

- **NOTE**: IT IS THIS TYPE OF IMBALANCE IN A CRANKSHAFT THAT CAN CAUSE DAMAGE TO THE MOUNT.

STABILITY OF TORQUE FREE MOTION

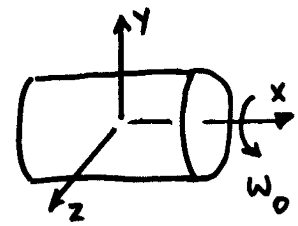
$\vec{M} \equiv 0$

- CAN GAIN A LOT OF INSIGHT BY CONSIDERING SPECIFIC TYPES OF MOTIONS, AND THEN SEEING HOW THE VEHICLE'S MOTION WOULD RESPOND TO SMALL PERTURBATIONS
  - MOTION "SIMILAR" TO INITIAL MOTION → STABLE

- CONSIDER ROTATION ABOUT ONE PRINCIPAL AXIS

$$\vec{\omega} = \omega_0 \vec{i} \quad \Rightarrow \quad \omega_B = \begin{bmatrix} \omega_0 \\ 0 \\ 0 \end{bmatrix}$$

X, Y, Z ~ BODY FRAME.



- NOW ADD A SLIGHT PERTURBATION TO THIS MOTION
  - ASSUME TORQUE FREE

$$\Rightarrow \quad \omega_B = \begin{bmatrix} \omega_0 + \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{bmatrix}$$

→ PERTURBATION TO ω_x WILL LEAD TO (HOPEFULLY) SMALL CHANGES TO ω_y, ω_z

⇒ NEED TO FIND A WAY TO PREDICT $\delta\omega_x(t)$, $\delta\omega_y(t)$, $\delta\omega_z(t)$

NOTE: PERTURBED MOTION MUST SATISFY EULER'S EQUATIONS.

- PERTURBED, TORQUE FREE EULER'S :

1) $0 = I_{xx}(\dot{\omega_0 + \delta\omega_x}) + (I_{zz} - I_{yy})\delta\omega_y \delta\omega_z$

2) $0 = I_{yy}\,\delta\dot{\omega}_y + (I_{xx} - I_{zz})(\omega_0 + \delta\omega_x)\delta\omega_z$

3) $0 = I_{zz}\,\delta\dot{\omega}_z + (I_{yy} - I_{xx})(\omega_0 + \delta\omega_x)\delta\omega_y$

- **KEY POINTS:**
  - ω_0 CONSTANT
  - $\omega_0 \gg \delta\omega_x, \delta\omega_y, \delta\omega_z$

- **LINEARIZE** E.O.M. BY DROPPING PRODUCTS OF $\delta\omega$'s

E.G. IN 1) $\cancel{\delta\omega_y \delta\omega_z} \rightarrow 0$

IN 2) $(\omega_0 + \delta\omega_x)\delta\omega_z = \omega_0 \delta\omega_z + \cancel{\delta\omega_x \delta\omega_z} \rightarrow 0$

- **LINEARIZED FORM:**

$\delta\dot{\omega}_x = 0 \longrightarrow \delta\omega_x$ CONSTANT

$$I_{yy}\,\delta\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_0\,\delta\omega_z = 0$$

$$I_{zz}\,\delta\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_0\,\delta\omega_y = 0$$

COMBINE: (DIFFERENTIATE FIRST ONE)

$$\delta\ddot{\omega}_y + \left[\frac{\omega_0^2 (I_{xx} - I_{yy})(I_{xx} - I_{zz})}{I_{yy} I_{zz}}\right]\delta\omega_y = 0$$

- DIFFERENTIAL EQUATION $\delta\ddot{\omega}_y + A\,\delta\omega_y = 0$

$$\Rightarrow \quad \delta\omega_y = B_1 e^{t\sqrt{-A}} + B_2 e^{-t\sqrt{-A}}$$

① IF $A > 0$, $-A < 0$ $\Rightarrow$ $\delta\omega_y$ SINUSOIDAL

$\Rightarrow$ $\delta\omega_y$, $\delta\omega_z$ TEND TO OSCILLATE

$\sim$ STABLE (NEUTRAL)

② IF $A < 0$, $-A > 0$ $\Rightarrow$ EXPONENTIAL GROWTH IN $\delta\omega_y \sim e^{\alpha t}$

$\therefore$ UNSTABLE

- IN OUR CASE $\boxed{A = \omega_0^2 \dfrac{(I_{xx} - I_{yy})(I_{xx} - I_{zz})}{I_{yy} I_{zz}}}$

FOR $A > 0$, NEED :

i) $I_{xx} > I_{yy}$ AND $I_{xx} > I_{zz}$

ii) $I_{yy} > I_{xx}$ AND $I_{zz} > I_{xx}$

STABLE CASES

(i) CORRESPONDS TO I_{xx} BEING LARGEST MOMENT OF INERTIA

(ii) CORRESPONDS TO I_{xx} BEING THE SMALLEST.

- RECALL THAT WE ARE SPINNING ABOUT X-AXIS.

- OBSERVATIONS:
  - IF INITIAL SPIN ABOUT AN INTERMEDIATE AXIS OF INERTIA ($I_{yy} > I_{xx} > I_{zz}$) THEN SPIN UNSTABLE.
  - SPIN ABOUT MAX/MIN AXES OF INERTIA ARE STABLE (ONLY "NEUTRAL")

$\boxed{\text{EXAMPLE}}$

- FURTHER THOUGHTS:
  - ROTATIONAL KINETIC ENERGY $T = \frac{1}{2} \omega_B^T I_B \omega_B$

$$\omega_B = \begin{bmatrix} \omega_0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad T_{ROT} = \frac{1}{2} I_{xx} \omega_0^2$$

$$H = I_{xx} \omega_0$$

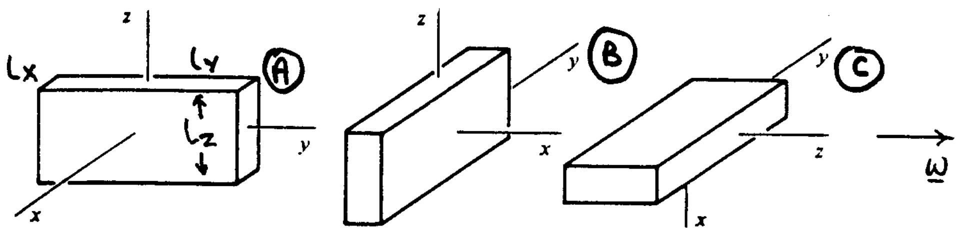
- WITH NO EXTERNAL MOMENTS, H FIXED

$$\Rightarrow T_{ROT} = \frac{1}{2} \frac{H^2}{I_{xx}} \propto \frac{1}{I_{xx}}$$

$\therefore$ IF I_{xx} <u>MINIMUM</u> INERTIA, THEN T_{ROT} IS THE <u>MAXIMUM</u> VALUE POSSIBLE.

$\Rightarrow$ $I_{xx} \sim$ MAXIMUM $\Rightarrow$ $T \sim$ MIN VALUE.

EXAMPLE: IF $l_x < l_z < l_y$, WHAT IS THE ORDERING OF I_x, I_y, I_z?



GW 534: $I_{xx} = \frac{m}{12}\left(l_y^2 + l_z^2\right)$; $I_{yy} = \frac{m}{12}\left(l_x^2 + l_z^2\right)$

$$I_{zz} = \frac{m}{12}\left(l_x^2 + l_y^2\right)$$

$$\Rightarrow \quad I_{yy} - I_{zz} = \frac{m}{12}\left(l_z^2 - l_y^2\right) < 0 \qquad \therefore \quad I_{yy} < I_{zz}$$

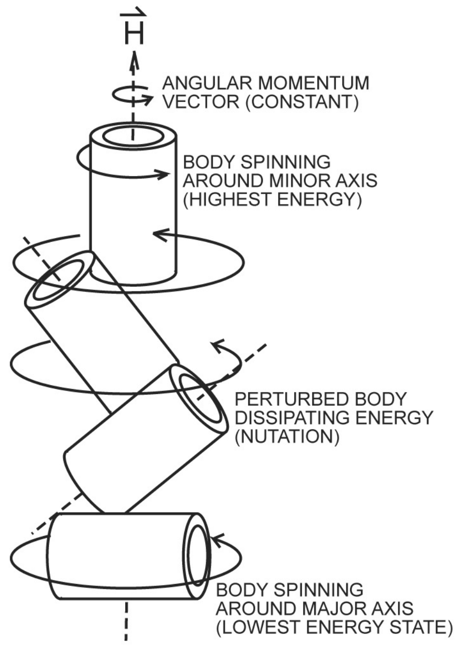
- CAN SHOW $I_{yy} < I_{zz} < I_{xx}$ — CONSISTENT WITH VISUAL "INSPECTION" OF MASS DISTRIBUTION
- SPIN STABILITY?

A   B   C

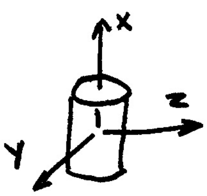
- MUCH MORE ON THIS TYPE OF PROBLEM LATER ON.

- SO, IF THERE IS AN INTERNAL ENERGY DISSIPATION MECHANISM IN THE SYSTEM, EXPECT T_{ROT} TO <u>REDUCE</u> $\Rightarrow$ ONLY ROTATIONS ABOUT THE MAXIMUM AXIS ARE STABLE

– SPIN ABOUT MIN AXIS <u>DEGENERATES</u>



FROM BRYSON.



RADIUS – a

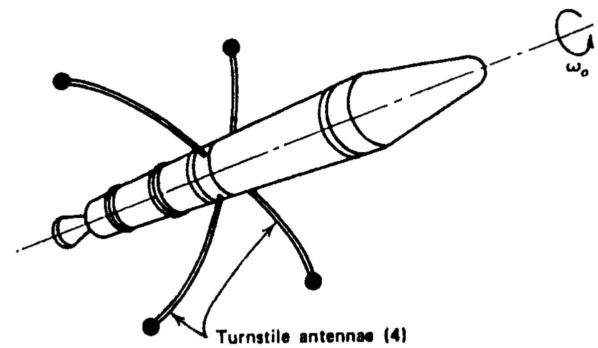
LENGTH – L

- CIRCULAR CYLINDER $\quad I_x \sim \frac{1}{2} m a^2$

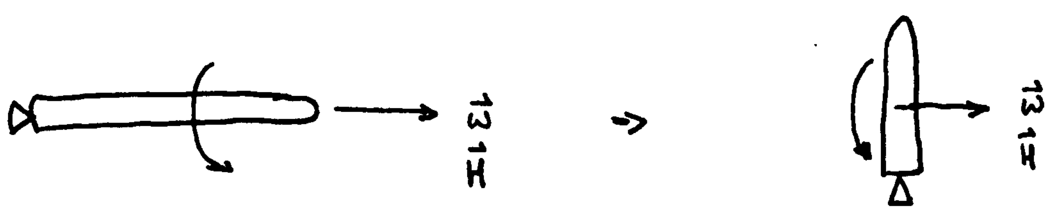
$$I_y, I_z \sim \frac{m}{12}\left(3a^2 + L^2\right)$$

$\therefore$ FOR LONG CYLINDER, $I_x < I_y = I_z$

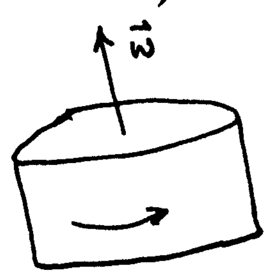
- INFAMOUS EXAMPLE: EXPLORER 1



- PLAN WAS TO SPIN SATELLITE ABOUT LONG AXIS ⇒ STABILITY (?) ⇒ MIN AXIS.
- BUT THE ANTENNAS DISSIPATED ENERGY
  - ⇒ MIN AXIS SPIN UNSTABLE
  - ⇒ BODY STARTED TO TUMBLE
  - ⇒ STABILIZED IN SPIN ABOUT MAJOR AXIS



- FOR STABLE SPIN, FLY A PLATE:



FURTHER INSIGHTS ON TORQUE FREE MOTION – GEOMETRIC

$\vec{M} = 0$

- TORQUE FREE – $\vec{H}$ CONSTANT
  - $|H_B|$ IS CONSTANT, BUT H_B CAN CHANGE.

- CAN ALSO SHOW THAT ROTATIONAL KINETIC ENERGY IS ALSO CONSTANT, I.E. $\dot{T}_{ROT} = 0$

WHY? $\quad T_{ROT} = \frac{1}{2}\vec{\omega}\cdot\vec{\vec{I}}\cdot\vec{\omega}$

$$\Rightarrow \dot{T}_{ROT} = \vec{\omega}\cdot\vec{\vec{I}}\cdot\dot{\vec{\omega}}^{I} + \vec{\omega}\cdot\dot{\vec{\vec{I}}}^{I}\cdot\vec{\omega}$$

BUT RECALL THAT $\dot{\vec{\omega}}^{B} \equiv \dot{\vec{\omega}}^{I}$; $\dot{\vec{\vec{I}}}^{I} = \vec{\omega}\times\vec{\vec{I}}$

$$\Rightarrow \dot{T}_{ROT} = \vec{\omega}\cdot\vec{\vec{I}}\cdot\dot{\vec{\omega}}^{B} + \vec{\omega}\cdot(\vec{\omega}\times\vec{\vec{I}})\cdot\vec{\omega}$$

BUT $\vec{\vec{I}}\cdot\dot{\vec{\omega}}^{B} = -\vec{\omega}\times(\vec{\vec{I}}\cdot\vec{\omega})$

$$\Rightarrow \dot{T}_{ROT} = \vec{\omega}\cdot\left(-\vec{\omega}\times(\vec{\vec{I}}\cdot\vec{\omega})\right) + \underbrace{\vec{\omega}\cdot(\vec{\omega}\times\vec{\vec{I}})\cdot\vec{\omega}}_{=0}$$

- RECALL THAT $\vec{\omega}\times\vec{A}$ PERPENDICULAR TO BOTH $\vec{\omega}$ AND $\vec{A}$, SO $\vec{\omega}\cdot(\vec{\omega}\times\vec{A}) \equiv 0$

$$\therefore \dot{T}_{ROT} = 0 \quad \Rightarrow \quad T_{ROT} \text{ CONSTANT.}$$

- ASSUMES THAT THERE ARE NO INTERNAL DISSIPATION MECHANISMS, AS DISCUSSED BEFORE.

- NOW ASSUME THAT THE BODY XYZ AXES ARE ALIGNED WITH THE PRINCIPAL AXES

$$\Rightarrow T_{ROT} = \frac{1}{2}\vec{\omega}\cdot\vec{H} = \frac{1}{2}\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}\begin{bmatrix} I_{xx} & & 0 \\ & I_{yy} & \\ 0 & & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix}$$

$$\Rightarrow 2T_{ROT} = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2$$

- THIS IS EQUIVALENT TO A CONSTRAINT EQUATION ON THE ALLOWABLE COMBINATIONS OF $\omega_x, \omega_y, \omega_z$ FOR A GIVEN T_{ROT}

  - SET OF POSSIBLE VALUES OF $\omega_x, \omega_y, \omega_z$ ARE <u>ON AN ELLIPSOID</u> ALIGNED WITH THE PRINCIPAL AXES.

- MORE ON 10-20

- ELLIPSOID SIZE IN EACH DIRECTION $\sim \sqrt{\dfrac{2T_{ROT}}{I_{KK}}}$

  SO LARGE $I_{KK} \rightarrow$ ELLIPSOID SMALL IN THAT DIRECTION.

- **MOMENTUM**: $\vec{H}$ FIXED, SO $|\vec{H}|^2$ MUST BE CONSTANT

$$H_B = \begin{bmatrix} I_{xx} & & 0 \\ & I_{yy} & \\ 0 & & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} I_{xx}\,\omega_x \\ I_{yy}\,\omega_y \\ I_{zz}\,\omega_z \end{bmatrix}$$

$$|H_B|^2 = I_{xx}^2\,\omega_x^2 + I_{yy}^2\,\omega_y^2 + I_{zz}^2\,\omega_z^2 = C$$

- PROVIDES ANOTHER CONSTRAINT ON THE POSSIBLE COMBINATIONS OF $\omega_x, \omega_y, \omega_z$

- MOTION OF THE BODY MUST SATISFY BOTH CONSTRAINTS

  → FEASIBLE ANGULAR VELOCITY COMBINATIONS $\omega_x, \omega_y, \omega_z$ AS SEEN IN THE BODY FRAME

  → MUST LIE AT THE INTERSECTION OF THE TWO ELLIPSOIDS.

    ⇒ CALLED A <u>POLHODE</u>

  → INTERSECTION CHANGES DEPENDING ON T_{ROT} AND $|H_B|$.

ENERGY

$$\frac{\omega_x^2}{2T/I_{xx}} + \frac{\omega_y^2}{2T/I_{yy}} + \frac{\omega_z^2}{2T/I_{zz}} = 1$$

MOMENTUM

$$\frac{\omega_x^2}{|H_B|^2/I_{xx}^2} + \frac{\omega_y^2}{|H_B|^2/I_{yy}^2} + \frac{\omega_z^2}{|H_B|^2/I_{zz}^2} = 1$$

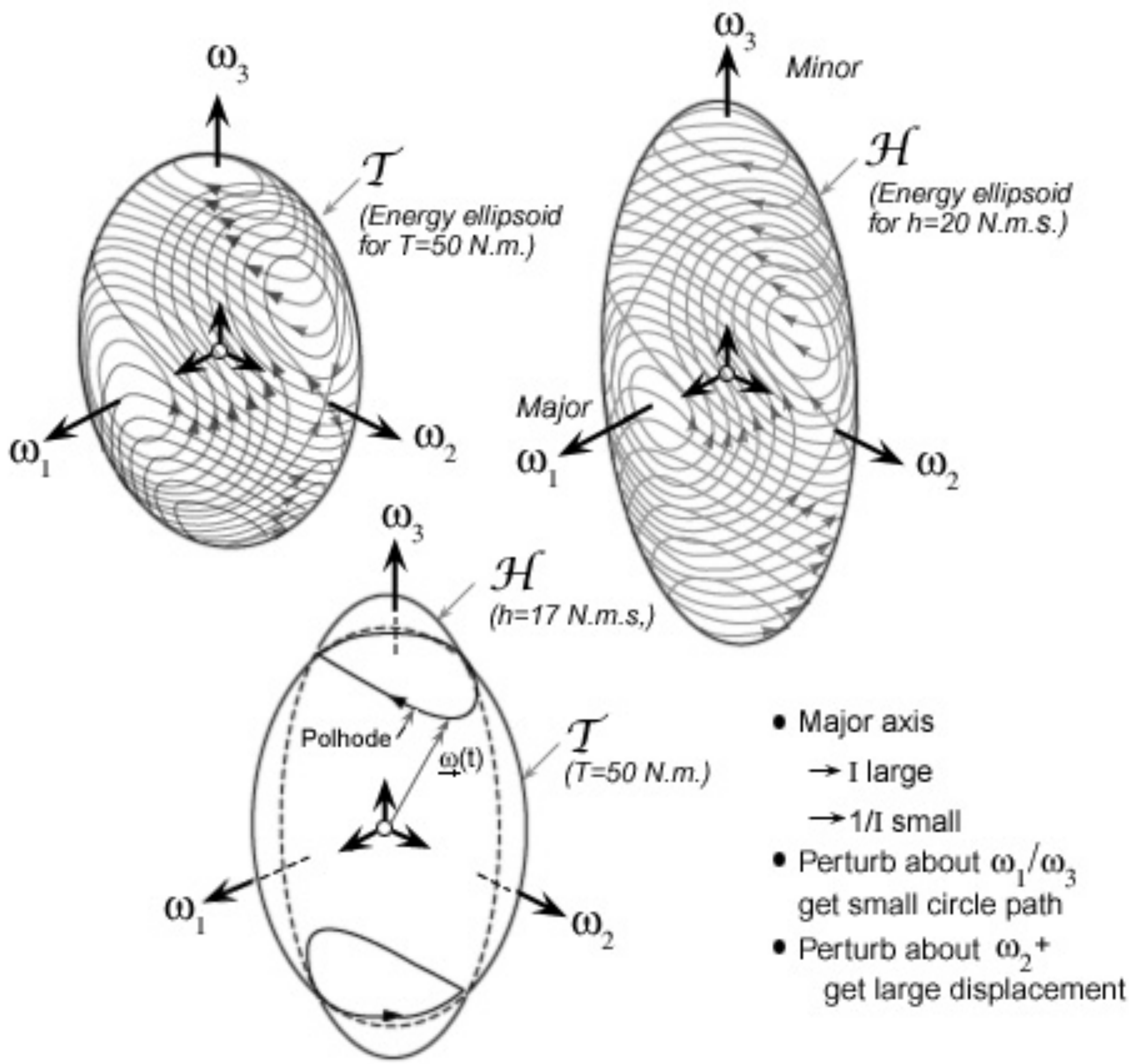
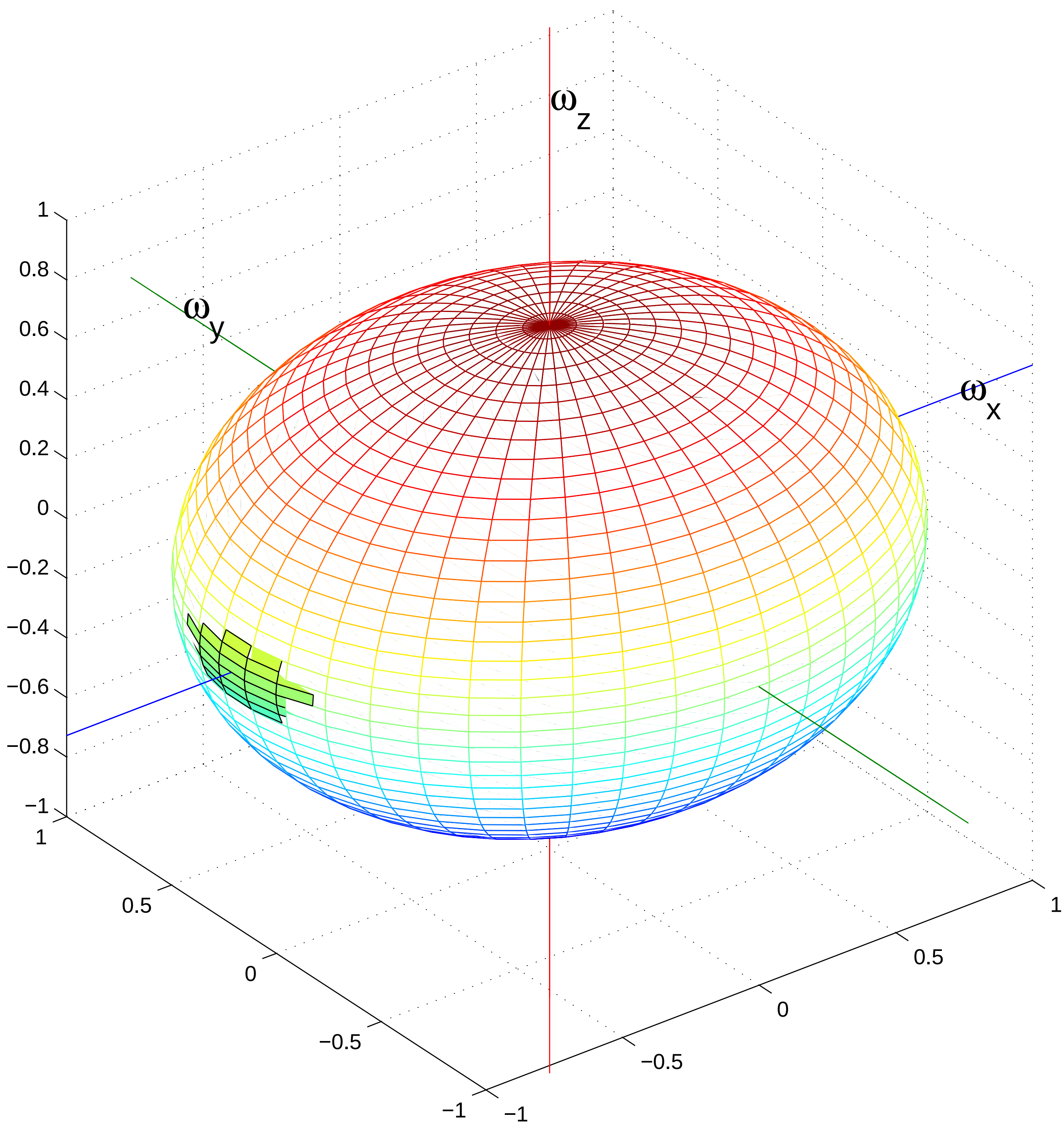
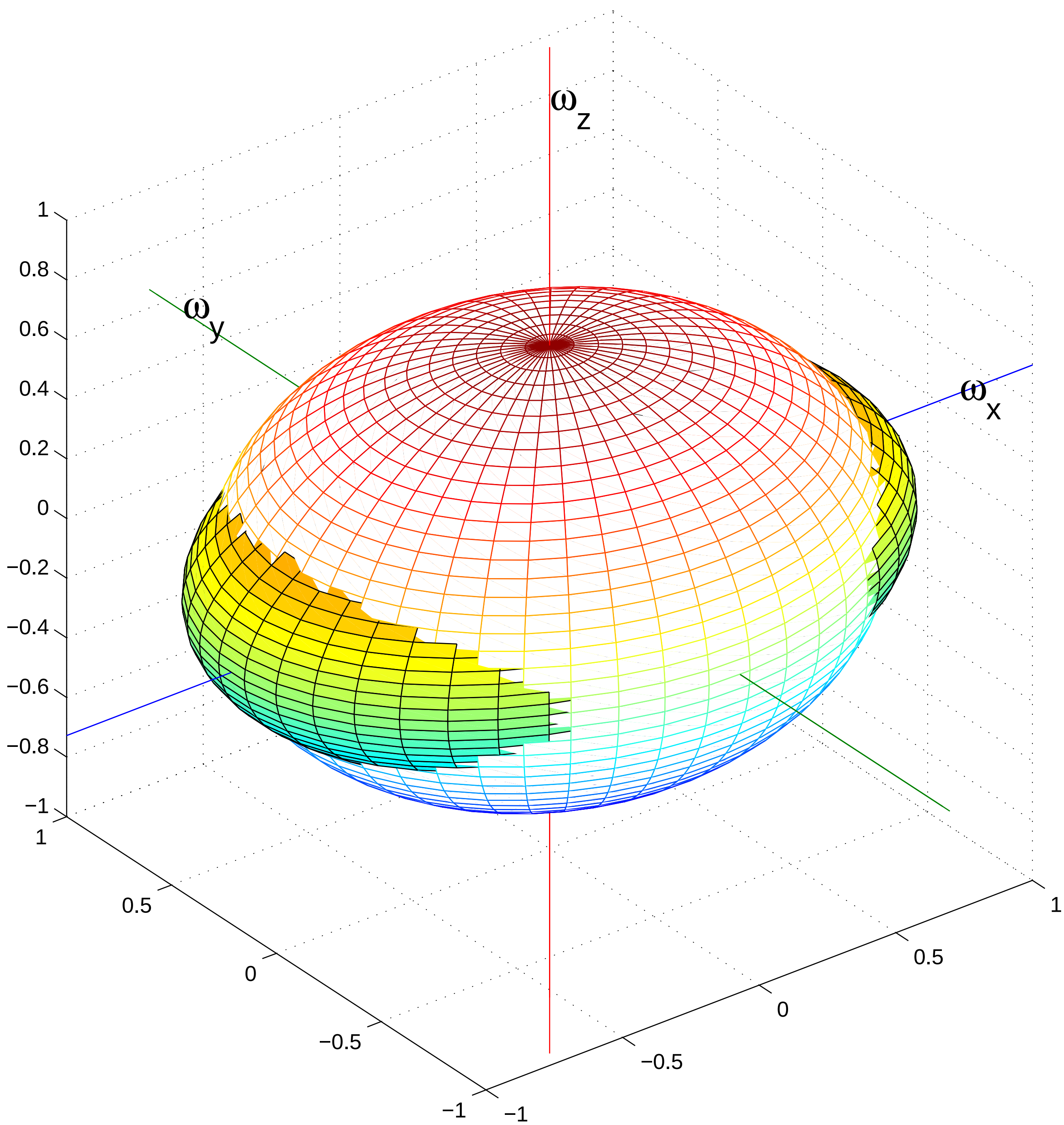


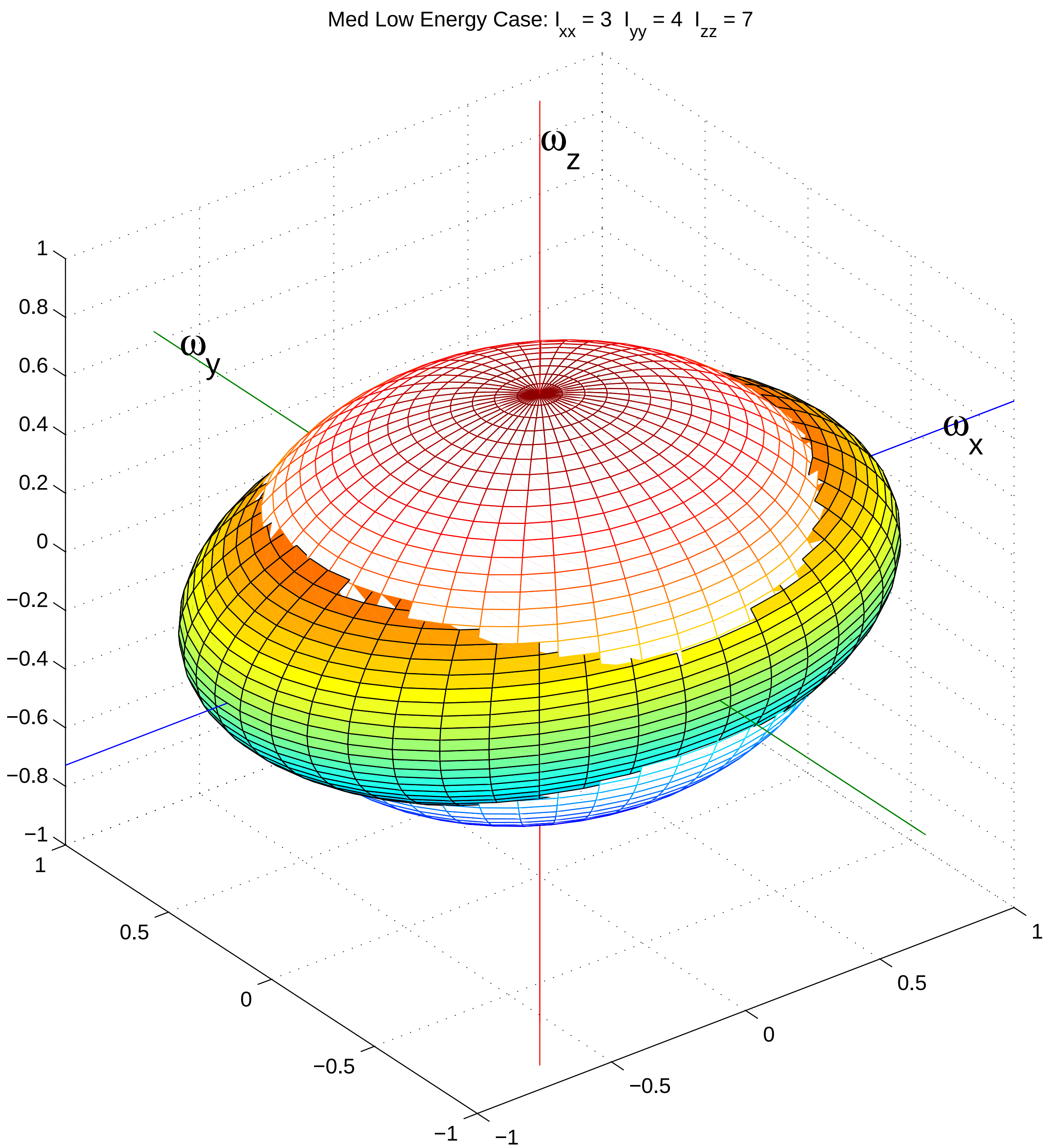
*Figure adapted from P.C. Hughes, **Spacecraft Attitude Dynamics** (John Wiley and Sons, 1986)*

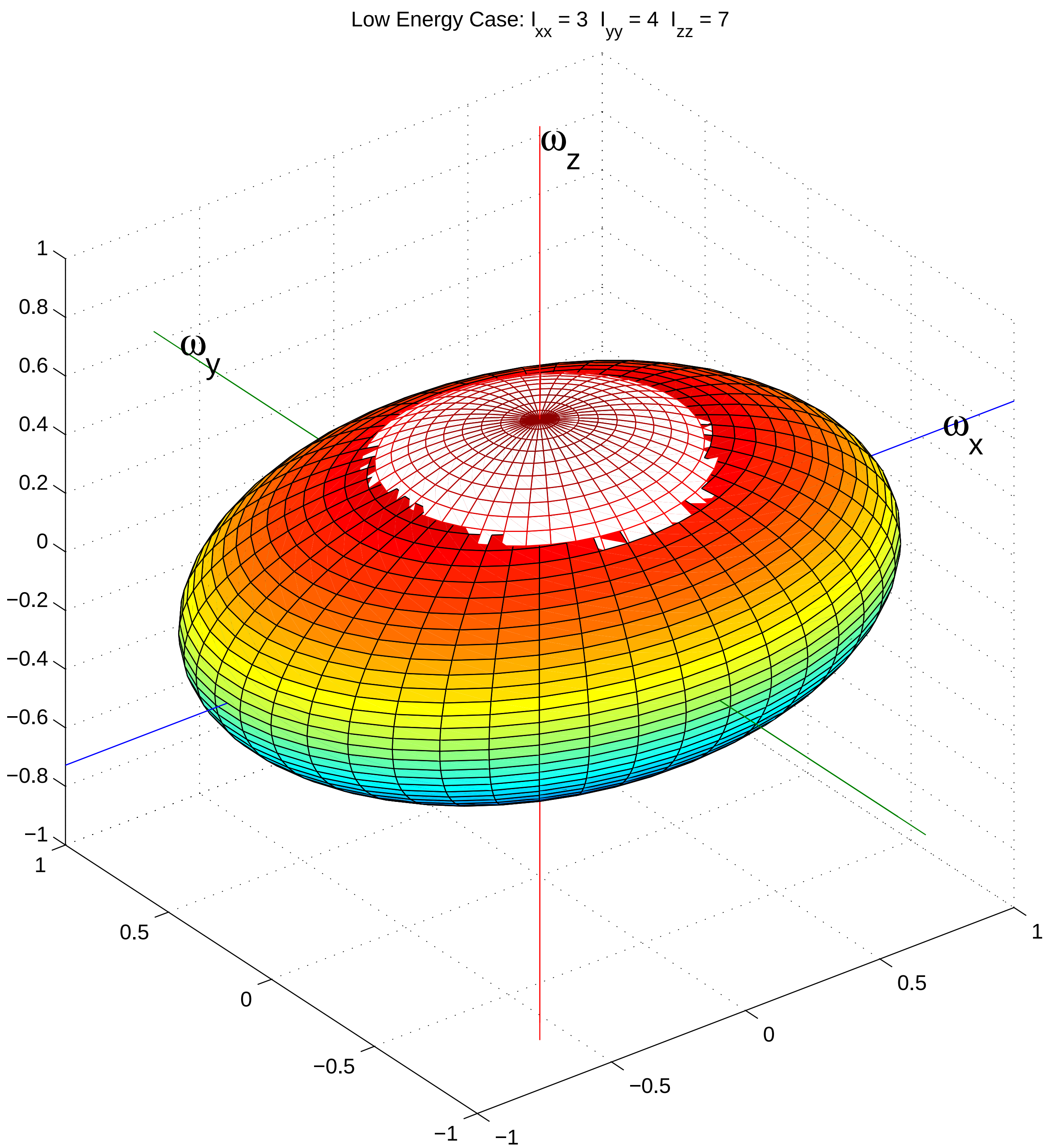
High Energy Case: $I_{xx} = 3$ $I_{yy} = 4$ $I_{zz} = 7$



Med High Energy Case: $I_{xx} = 3$ $I_{yy} = 4$ $I_{zz} = 7$




Med Low Energy Case: I_{xx} = 3 I_{yy} = 4 I_{zz} = 7
ω_z
ω_y
ω_x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1



Low Energy Case: I_{xx} = 3 I_{yy} = 4 I_{zz} = 7
ω_z
ω_y
ω_x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1