LECTURE # 11

KINEMATICS OF RIGID BODIES

- INERTIA MATRIX AND DYADIC
- $\vec{H}$ CALCULATION , T CALCULATION
- PRINCIPAL AXES AND ROTATIONS

RIGID BODY DYNAMICS

- TWO COMPONENTS TO RIGID BODY MOTION:

  TRANSLATIONAL $\vec{F} = m\ddot{\vec{r}}_{cm}^{\,I}$

  ROTATIONAL $\vec{M} = \dot{\vec{H}}^{I}$

  - DECOUPLE PROVIDED $\vec{F}$ IND OF ROTATION AND $\vec{M}$ IND OF TRANSLATION.

  → CAN TREAT THE COMPLEX MOTION OF A SYSTEM AS A:

  1. POINT MASS MOVING AS THE CENTER OF MASS

     \+

  2. BODY ROTATION ABOUT THE CENTER OF MASS.

- ALREADY STUDIED CASE 1 IN DEPTH

  ⇒ CONSIDER CASE 2 FOR GENERAL 3D MOTION.

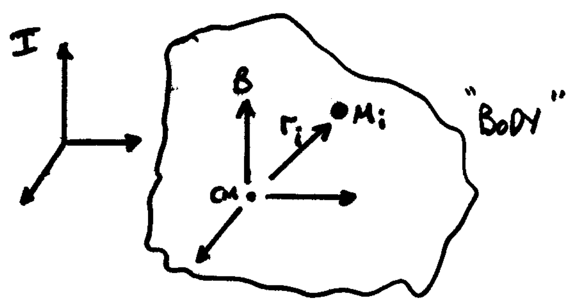
- **QUICK REVIEW**

  - ANGULAR MOMENTUM OF A PARTICLE i ABOUT THE CENTER OF MASS IS EQUAL TO THE MOMENT OF THE PARTICLE'S LINEAR MOMENTUM ABOUT THE C.O.M. (NOT NECESSARY, BUT SIMPLIFIES).

$$\vec{H}_i = \vec{r}_i \times (m_i \vec{v}_i) \quad \overset{\text{BODY}}{\Rightarrow} \quad \vec{H} = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

$\vec{r}_i$: LOCATION OF PARTICLE WRT C.O.M.

$\vec{v}_i$: ABSOLUTE VELOCITY OF PARTICLE i



- FOR RIGID BODY WITH CONTINUOUS MASS DISTRIBUTION

$$\{\text{PARTICLE } m_i\} \Rightarrow \left\{\begin{matrix}\text{MASS } dm \text{ OF SMALL} \\ \text{VOLUME } dV\end{matrix}\right\}$$

$$\vec{H} = \int_B \vec{r} \times \vec{v}\, dm$$

- USE TRANSPORT THEOREM

$$\vec{v} = \vec{V}_{COM} + \vec{\omega} \times \vec{r}$$

- SUBSTITUTE TO GET:

$$\vec{H} = \int_B \vec{r} \times \cancel{\vec{v}_{COM}}^{\,0}\, dm + \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

SINCE V_{COM} CONSTANT

AND ORIGIN OF $\vec{r}$ IS

C.O.M.

$$\Rightarrow \quad \vec{H} = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

INERTIA DEFINITIONS

- EXPANSION OF THE INERTIA
  - RB, REF POINT AT ORIGIN OF CARTESIAN COORDINATE SYSTEM
  - VOLUME ELEMENT AT $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
  - ANGULAR VELOCITY OF BODY IN TERMS OF CARTESIAN COMPONENTS:

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

⇒ NOW EXPAND $\vec{r} \times (\vec{\omega} \times \vec{r})$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \begin{array}{l} (z\omega_y - y\omega_z)\,\vec{i} \\ + (x\omega_z - z\omega_x)\,\vec{j} \\ + (y\omega_x - x\omega_y)\,\vec{k} \end{array}$$

So

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ z\omega_y - y\omega_z & x\omega_z - z\omega_x & y\omega_x - x\omega_y \end{vmatrix}$$

$$= \left[(y^2+z^2)\,\omega_x - xy\,\omega_y - xz\,\omega_z\right]\vec{i}$$

$$+ \left[-yx\,\omega_x + (x^2+z^2)\,\omega_y - yz\,\omega_z\right]\vec{j}$$

$$+ \left[-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\,\omega_z\right]\vec{k}$$

- DEFINE THE MOMENTS OF INERTIA AS:

$$I_{xx} = \int_B (y^2 + z^2)\, dm \quad ; \quad I_{xy} = I_{yx} = -\int_B xy\, dm$$

$$I_{yy} = \int_B (x^2 + z^2)\, dm \quad ; \quad I_{xz} = I_{zx} = -\int_B xz\, dm$$

$$I_{zz} = \int_B (x^2 + y^2)\, dm \quad ; \quad I_{yz} = I_{zy} = -\int_B yz\, dm$$

- THEN

$$H = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

$$= [I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z]\, \vec{i}$$

$$+ [I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z]\, \vec{j}$$

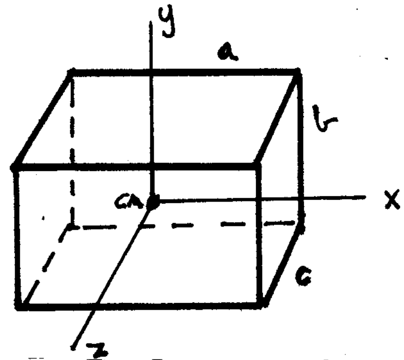
$$+ [I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z]\, \vec{k}$$

$$\equiv H_x \vec{i} + H_y \vec{j} + H_z \vec{k}$$

- FINDING $I_{xx}, I_{xz}, I_{xy}, \ldots$ REQUIRES MANY TRIPLE INTEGRALS
- MATRIX NOTATION:

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\text{INERTIA MATRIX}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

TYPICAL EXAMPLE: BOX $(a \times b \times c)$



- FIND I_{xx} AT C.O.M

$$I_{xx} = \int_B (y^2+z^2)\,dm = \int_{-\frac{c}{2}}^{\frac{c}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-a/2}^{a/2} \rho\,(y^2+z^2)\,dx\,dy\,dz$$

$$= \rho a \left[\int_{-c/2}^{c/2} z^2\,dz + \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\,dy\right]$$

$$= \rho a \left[\left(\frac{z^3}{3}\right)_{-c/2}^{c/2} + \left(\frac{y^3}{3}\right)_{-\frac{b}{2}}^{\frac{b}{2}}\right]$$

$$= \frac{\rho abc}{12}\left(b^2+c^2\right) \qquad m = \rho abc$$

$$\therefore\ I_{xx} = \frac{m}{12}\left(b^2+c^2\right)$$

- MANY OTHER EXAMPLES IN THE TEXTBOOKS.

KEY POINTS:

1) FOR PLANAR BODIES WITH ORIGIN IN THE PLANE (XY)

$$I_{xz} = I_{yz} = 0$$

$$I_{zz} = I_{xx} + I_{yy}$$

2) FOR 3-D BODIES WITH A PLANE OF SYMMETRY, THE CROSS MOMENTS OF INERTIA ACROSS THE PLANE ARE ZERO

- PLANE OF SYMMETRY X-Y

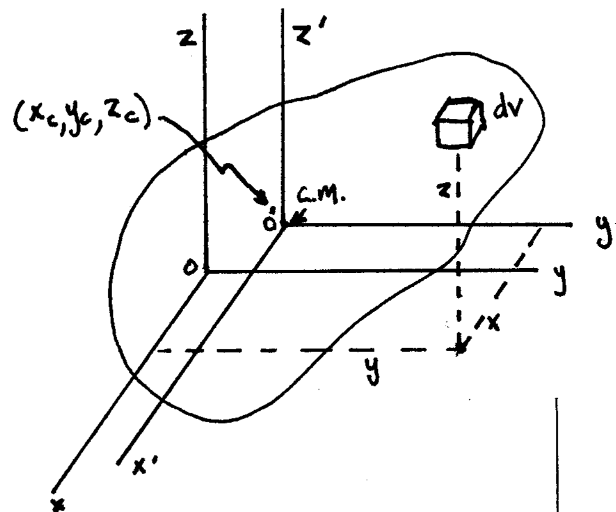
$$\Rightarrow \quad I_{xz} = I_{yz} = 0$$

- "MASS EVENLY DISTRIBUTED ON BOTH SIDES OF THE PLANE"

3) IF, FURTHERMORE, ONE OF THE COORDINATE AXES IS THE SYMMETRY AXIS OF A BODY OF REVOLUTION, THEN ALL CROSS MOMENTS OF INERTIA ARE ZERO.

TRANSLATION OF COORDINATES

- OFTEN HAVE THE INERTIAS ABOUT ONE SET OF AXES AND NEED IT ABOUT A SECOND SET:
  - PARALLEL TO FIRST
  - OFFSET



$$X = X' + X_c$$
$$Y = Y' + Y_c$$
$$Z = Z' + Z_c$$

- RESULT IS THE PARALLEL AXIS THM.

$$I_{KK} = I_{K'K'} + md^2$$

WHERE d IS THE DISTANCE BETWEEN A GIVEN PRIMED AND UNPRIMED AXIS

- SO:

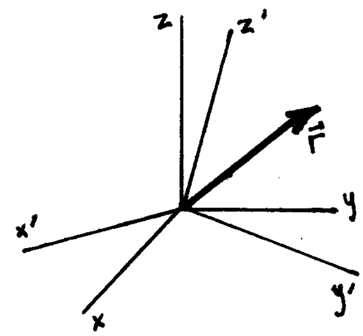
$$I_{xx} = I_{x'x'} + m(y_c^2 + z_c^2)$$
$$I_{yy} = I_{y'y'} + m(x_c^2 + z_c^2)$$
$$I_{zz} = I_{z'z'} + m(x_c^2 + y_c^2)$$

$$I_{xy} = I_{x'y'} - m x_c y_c \; ; \; I_{xz} = I_{x'z'} - m x_c z_c$$
$$I_{yz} = I_{y'z'} - m y_c z_c$$

ROTATION OF COORDINATES.

- ON 10-3B, INTRODUCED THE INERTIA MATRIX

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yz} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$



FOR THE X-Y-Z COORDINATE SYSTEM (F1) (ORIGIN AT C.O.M)

- WHAT IF WE HAVE A SECOND FRAME (F2) X'-Y'-Z' (SAME ORIGIN) THAT IS REACHED FROM THE FIRST THROUGH A GENERAL ROTATION "R_{21}" (SEE 2-10)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_{21} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

RECALL THAT $R_{21}^{-1} \equiv R_{21}^{T}$

- IN FRAME 1 $\qquad H_1 = I_1 \omega_1$

- GIVEN H_1, FIND $\quad H_2 = R_{21} H_1$

  " $\quad \omega_1 \quad$ " $\quad \omega_2 = R_{21} \omega_1$

$$\therefore \; H_2 = I_2 \omega_2 \Rightarrow R_{21} H_1 = R_{21} I_1 \omega_1 = R_{21} I_1 R_{21}^{-1} \omega_2$$

$$\therefore \; I_2 = R_{21} I_1 R_{21}^{T}$$

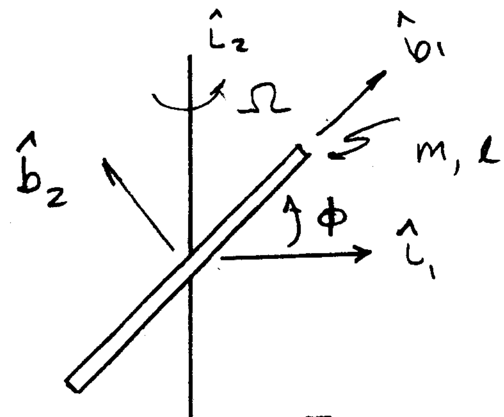
- SO TO ROTATE THE INERTIA MATRIX, WE NEED TO PRE- AND POST- MUTIPLY BY R_{21}

$$I_2 = R_{21} I_1 R_{21}^T$$

$$I_1 = R_{21}^T I_2 R_{21}$$

EXAMPLE: ROD ATTACH TO SHAFT SPINS AT RATE Ω, FIND $\vec{H}$ IN INERTIAL FRAME COORDINATES.

$$\vec{\omega} = \Omega \, \hat{i}_2$$



KEY POINT: INERTIAS EASY TO FIND USING $\hat{b}$ COORDINATES:

$$I_{b_2 b_2} = I_{b_3 b_3} = \frac{ml^2}{12} \qquad I_b = \frac{ml^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{bi} = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_i = R_{bi}^T I_b R_{bi} = \frac{ml^2}{12}\begin{bmatrix} +\sin^2\phi & -\sin\phi\cos\phi & 0 \\ -\cos\phi\sin\phi & \cos^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow H = \frac{ml^2\Omega}{12}\begin{bmatrix} -\sin\phi\cos\phi \\ \cos^2\phi \\ 0 \end{bmatrix}$$

PRINCIPAL AXES OF INERTIA

- IN GENERAL THE INERTIA MATRIX IS FULLY POPULATED

  ⇒ BUT CAN ALWAYS FIND A NEW FRAME (REACHED BY A ROTATION) FOR WHICH THE INERTIA MATRIX IS DIAGONAL

$$I \Rightarrow I' = \begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix}$$

- $I_{x'x'}, I_{y'y'}, I_{z'z'}$ CALLED PRINCIPAL MOMENTS OF INERTIA
- x', y', z' CALLED PRINCIPAL AXES

- DIAGONAL I MAKES THINGS MUCH EASIER:

$$H = I\omega \Rightarrow H_x = I_{xx}\,\omega_x\,;\ H_y = I_{yy}\,\omega_y\,;\ \ldots$$

$$T = \frac{1}{2}\omega^T I \omega \Rightarrow T = \frac{1}{2}\sum_{i=1}^{3} I_{ii}\,\omega_i^2$$

- GREAT, BUT HOW FIND THESE PRINCIPAL AXES?
  $\Rightarrow$ EIGENVALUE PROBLEM.

- GIVEN SYMMETRIC MATRIX A (3x3)
  FIND EIGENVALUES λ_i AND EIGENVECTORS v_i

$$A v_i = \lambda_i v_i \qquad i = 1, \ldots, 3 \qquad v_i^T v_j = 0 \quad \forall i,j \;\; i \neq j$$

$$\therefore \; A \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

OR $\quad AV = V\Lambda$

$\Rightarrow \quad A = V \Lambda V^{-1}$ BUT GIVEN PROPERTIES OF v_i, $V^{-1} = V^T$

$$\therefore \; A = V \Lambda V^T \; ; \quad \Lambda = V^T A V$$

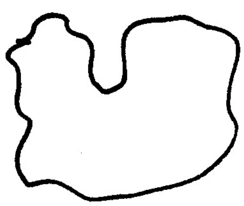
$\Rightarrow$ V ACTS AS A ROTATION MATRIX ON "A" YIELDING A DIAGONAL Λ

- GIVEN I, PERFORM EV, $\vec{EV}$ DECOMPOSITION
  TO OBTAIN
  - EIGENVALUES ($I_{x'x'}, I_{y'y'}, I_{z'z'}$)
  - EIGENVECTORS (ROTATION MATRIX)

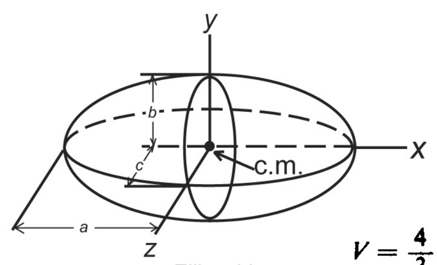
- IN THE PRINCIPAL AXES, THE INERTIA MATRIX IS DIAGONAL

  ⇒ COULD JUST CONSIDER THE BODY NOW AS THE EQUIVALENT, BUT SIMPLER, SHAPE THAT GIVES THE SAME PRINCIPAL MOMENTS OF INERTIA (AND MASS)

⇒ "POTATO"



⇒



Ellipsoid

$$V = \frac{4}{3}\pi abc$$

$$I_{xx} = \frac{m}{5}(b^2 + c^2)$$

$$I_{yy} = \frac{m}{5}(a^2 + c^2)$$

$$I_{zz} = \frac{m}{5}(a^2 + b^2)$$

- SAME PRINCIPAL MOMENTS OF INERTIA AND MASSES ⇒ TWO BODIES ARE DYNAMICALLY EQUIVALENT.

DYADIC NOTATION

- COULD WORK WITH THE MATRIX NOTATION, BUT THIS IS A BIT CLUMSY WHEN DOING THE DERIVATIVES

  ⇒ MORE CONVENIENT TO USE VECTOR NOTATION

- OK, BUT HOW DO WE WRITE $\vec{H}$ IN TERMS OF THE INERTIAS AND ANGULAR VELOCITY?

  ⇒ NEED TO INTRODUCE A NEW ENTITY CALLED THE <u>INERTIA</u> <u>DYADIC</u> $\vec{\vec{I}}$

$$\vec{H} = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm \quad \Rightarrow \quad \vec{H} = \vec{\vec{I}} \cdot \vec{\omega}$$

- CAN ALSO SHOW THAT:

$$T = \frac{1}{2}\,\vec{\omega} \cdot \vec{H} = \frac{1}{2}\,\vec{\omega} \cdot \vec{\vec{I}} \cdot \vec{\omega}$$

DYADICS

- "EVERYTHING YOU NEED TO KNOW" ABOUT DYADICS.

  - FOR NOW.

- LET $\vec{A}, \vec{B}$ BE 2 VECTORS, THEN THEIR <u>DOT</u> <u>PRODUCT</u> IS A SCALAR.

$$\vec{A} \cdot \vec{B} = \left(a_x \vec{i} + a_y \vec{j} + a_z \vec{k}\right) \cdot \left(b_x \vec{i} + b_y \vec{j} + b_z \vec{k}\right)$$

$$= a_x b_x + a_y b_y + a_z b_z$$

WHY? – PROBABLY NEVER THOUGHT ABOUT IT

– BUT THE RULES ARE: $\begin{cases} \vec{i} \cdot \vec{i} = 1 \\ \vec{i} \cdot \vec{j} = 0 \\ \vec{i} \cdot \vec{k} = 0 \end{cases}$

$\Rightarrow$ SO ONLY THE $\vec{i} \cdot \vec{i}$; $\vec{j} \cdot \vec{j}$; $\vec{k} \cdot \vec{k}$ ARE NON-ZERO.

- A <u>DYAD</u> IS LIKE A SECOND ORDER VECTOR "$\vec{i}\vec{i}$"

  – DOT PRODUCT OF A DYAD AND A VECTOR IS <u>STILL</u> A VECTOR.

$\vec{\vec{I}} \cdot \vec{w}$

E.G. WILL GET TERMS LIKE:

$$I_{xx} \vec{i}\vec{i} \cdot \left(w_x \vec{i} + w_y \vec{j} + w_z \vec{k}\right)$$

$$= I_{xx} w_x \vec{i}$$

SINCE $\begin{cases} \vec{i}\vec{i} \cdot \vec{i} = \vec{i} \\ \vec{i}\vec{i} \cdot \vec{j} = 0 \\ \vec{i}\vec{i} \cdot \vec{k} = 0 \end{cases}$

- SO, CAN SHOW THAT THE INERTIA DYAD IS OF THE FORM

$$\vec{\vec{I}} = I_{xx}\,\vec{i}\,\vec{i} + I_{xy}\,\vec{i}\,\vec{j} + I_{xz}\,\vec{i}\,\vec{k}$$
$$+ I_{yx}\,\vec{j}\,\vec{i} + I_{yy}\,\vec{j}\,\vec{j} + I_{yz}\,\vec{j}\,\vec{k}$$
$$+ I_{zx}\,\vec{k}\,\vec{i} + I_{zy}\,\vec{k}\,\vec{j} + I_{zz}\,\vec{k}\,\vec{k}$$

– $I_{xx}, I_{xz}, \ldots$ AS DEFINED PREVIOUSLY. (7-4)

- THUS, $\vec{H} = \vec{\vec{I}} \cdot \vec{\omega} = \vec{i}\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)$

$$+ \vec{j}\left(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)$$
$$+ \vec{k}\left(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\right)$$

- FORM FOR $\vec{\vec{I}}$ ARISES BECAUSE YOU CAN SHOW (10-6)

$$\vec{\vec{I}} = \int_B \left[(\vec{r}\cdot\vec{r})\vec{\vec{U}} - \vec{r}\,\vec{r} \right] dm$$

$\vec{\vec{U}} = \vec{i}\,\vec{i} + \vec{j}\,\vec{j} + \vec{k}\,\vec{k}$ UNIT DYADIC

MORE DETAILS ON DYADIC NOTATION

- DYADIC NOTATION – MOTIVATION

$$\vec{H} = \int_B \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

$\Rightarrow$ LOOK AT THE INTEGRAND: $\vec{r} \times (\vec{\omega} \times \vec{r})$

– USE THE VECTOR TRIPLE PRODUCT

$$(\vec{A} \times (\vec{B} \times \vec{C})) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

$$\Rightarrow \quad \vec{r} \times (\vec{\omega} \times \vec{r}) = (\vec{r} \cdot \vec{r})\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}$$

$$= (\vec{r} \cdot \vec{r})\vec{\omega} - \vec{r}(\vec{r} \cdot \vec{\omega})$$

$$= (\vec{r} \cdot \vec{r})\vec{\omega} - \vec{r}\vec{r} \cdot \vec{\omega}$$

– BY DEFINITION: $\vec{\omega} = \vec{\vec{U}} \cdot \vec{\omega}$; $\vec{\vec{U}} \equiv \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}$

$$\therefore \quad \vec{r} \times (\vec{\omega} \times \vec{r}) = \left[(\vec{r} \cdot \vec{r})\vec{\vec{U}} - \vec{r}\vec{r}\right] \cdot \vec{\omega}$$

$$\therefore \quad \vec{H} = \int_B \left[(\vec{r} \cdot \vec{r})\vec{\vec{U}} - \vec{r}\vec{r}\right] dm \cdot \vec{\omega}$$

$$\vec{H} = \vec{\vec{I}} \cdot \vec{\omega}$$

KINETIC ENERGY

- SHOWED THAT TOTAL KINETIC ENERGY FOR A SYSTEM OF N PARTICLES:

$$T = \frac{1}{2} m V_c^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left| \dot{\vec{\rho}}_i^{\,I} \right|^2$$

V_c - SPEED OF C.O.M.

$\dot{\vec{\rho}}_i^{\,I}$ - VELOCITY OF i^{TH} PARTICLE WRT C.O.M.

- FOR A RIGID BODY, ONLY ALLOWED MOTION WRT C.O.M ARE DUE TO ROTATIONS: $\dot{\vec{\rho}}_i^{\,I} \equiv \vec{\omega} \times \vec{\rho}_i$

$$\Rightarrow \left| \dot{\vec{\rho}}_i^{\,I} \right|^2 = \dot{\vec{\rho}}_i^{\,I} \cdot \dot{\vec{\rho}}_i^{\,I} = \dot{\vec{\rho}}_i^{\,I} \cdot \left(\vec{\omega} \times \vec{\rho}_i \right)$$

- DEFINE $\quad T_{ROT} = \frac{1}{2} \sum_{i=1}^{N} m_i \left| \dot{\vec{\rho}}_i^{\,I} \right|^2$

FOR CTS BODY $\Rightarrow T_{ROT} = \frac{1}{2} \int_B \vec{\omega} \cdot \left(\vec{\rho} \times \dot{\vec{\rho}}^{\,I} \right) dm$

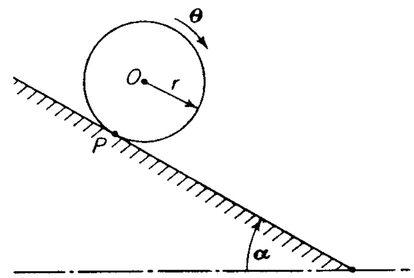
$$= \frac{\vec{\omega}}{2} \cdot \underbrace{\int_B \left(\vec{\rho} \times \dot{\vec{\rho}}^{\,I} \right) dm}_{\vec{H}_{com}}$$

$$\therefore \quad T_{ROT} = \frac{1}{2} \vec{\omega} \cdot \vec{H}_{com} = \frac{1}{2} \vec{\omega} \cdot \vec{\vec{I}} \cdot \vec{\omega}$$

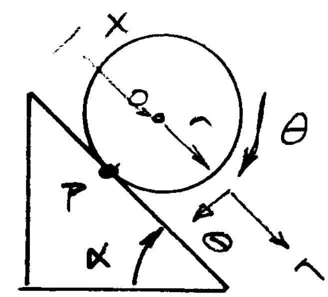
EXAMPLE: RIGID BODY MOTION IN A PLANE

GW 358

- UNIFORM DISC RADIUS r MASS m ROLLS DOWN RAMP W/O SLIPPING.



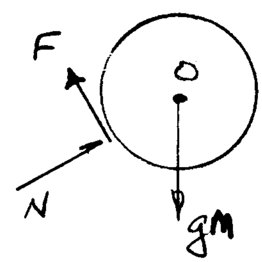
- $\vec{M} = \dot{\vec{H}}^{I}$



$\vec{\omega}$ INTO PAGE. ($\vec{z}$ DIRECTION)

DUE TO SYMMETRY, $\vec{z}$ A PRINCIPAL AXIS

$\Rightarrow$ $\vec{H}$ AND $\dot{\vec{H}}^{I}$ ARE PARALLEL



- TAKE REF. POINT AT O

$\Rightarrow$ MOMENT OF INERTIA $I_o = \frac{1}{2} m r^2$

$$H_z = I_o \dot{\theta} \quad , \quad \dot{H}_z^{I} = I_o \ddot{\theta}$$

- FBD $\Rightarrow$ $M_o = Fr$ $\Rightarrow$ $F = \frac{1}{2} m r \ddot{\theta}$

- SUM FORCES ALONG RAMP $mg \sin\alpha - F = m r \ddot{\theta}$

$$\therefore \quad mg \sin\alpha = \frac{3}{2} m r \ddot{\theta} \; ; \quad \ddot{\theta} = \frac{2}{3} \frac{g \sin\alpha}{r}$$

- ALTERNATIVE : LAGRANGE

$$T = \frac{1}{2} m v_o^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m (r\dot{\theta})^2 + \frac{1}{4} m r^2 \dot{\theta}^2 = \frac{3}{4} m r^2 \dot{\theta}^2$$

$$V = -mg(r\theta \sin\alpha)$$ HEIGHT DOWN FROM TOP

$$L = T - V = \frac{3}{4} m r^2 \dot{\theta}^2 + mgr\theta \sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{3}{2} m r^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mgr \sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{3}{2} m r^2 \ddot{\theta} - mgr \sin\alpha$$

$$\Rightarrow \ddot{\theta} = \frac{2}{3} \frac{g}{r} \sin\alpha$$