Hyperbolic Equations : Scalar One-Dimensional Conservation Laws

Lecture 11

1 Scalar Conservation Laws

1.1 Definitions

1.1.1 Conservative Form

SLIDE 1

General form (1D):

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$u(x,t)$: is the unknown conserved quantity
(mass, momentum, heat, ...)

$f(u)$: is the flux

1.1.2 Primitive Form

SLIDE 2

Can also be written ...

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial u}{\partial t} + \frac{df}{du}\frac{\partial u}{\partial x} = 0$$

$$\boxed{\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0}$$

where $a(u) = \dfrac{df}{du}$.

N1

Note 1 ***More General Conservation Laws***

In some applications, the flux function f may depend explicitly (not through u) on x; i.e. $f(u,x)$. In such cases, the primitive form of the equation becomes

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = g(u)$$

where $a(u) = \frac{\partial f}{\partial u}$; and $g(u) = -\frac{\partial f}{\partial x}$ plays the role of a source term.
The procedures presented here will be generally applicable, sometimes with small modifications, to this more general form. However, for clarity of presentation we will restrict ourselves to the case where f can be determined once u is known.

1.1.3 Integral Form

SLIDE 3

Consider a *fixed* domain $\Omega \equiv [x_L, x_R] \in \mathbb{R}$

$$\int_\Omega \left(\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \right) \, dV = 0$$

$$\boxed{\frac{d}{dt} \int_\Omega u \, dV = -[f(u_R) - f(u_L)]}$$

The integral form is the most general form of the conservation law from which the differential forms are derived. In contrast with the differential form, we note that the integral from is well defined even when the solution u and/or the flux f are discontinuous. We show below, an example of derivation of the different forms of the conservation laws from physical principles.

1.2 Derivation Example

1.2.1 Conservation of Mass

SLIDE 4

Consider a volume Ω enclosed by surface $\partial\Omega$ containing fluid of density $\rho(\boldsymbol{x}, t)$ and known velocity $\boldsymbol{v}(\boldsymbol{x}, t)$

SLIDE 5

$$\begin{array}{ccc} \text{RATE OF CHANGE OF MASS INSIDE } \Omega & \equiv & \text{MASS FLUX OF FLUID THROUGH } \partial\Omega \\ \dfrac{\partial}{\partial t} \displaystyle\int_\Omega \rho \, dV & = & -\displaystyle\int_{\partial\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{n} \, dS \\ & = & -\displaystyle\int_\Omega \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \, dV \end{array}$$

$$\int_\Omega \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \right] \, dV = 0$$

holds for all Ω, so we can write

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0}$$

This is the **differential form** of the conservation law.

To derive the differential form of the conservation law, we have assumed that $\rho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ are differentiable functions.

1.3 Examples

1.3.1 Linear Advection Equation

SLIDE 6

Model convection of a concentration $\rho(x,t)$:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial \rho a}{\partial x} = \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0}$$

a : constant

N2

Note 2 ***Advection-Diffusion Equation***

Consider the flux of a chemical past some point in a stream. If there is no diffusion in the flow, the concentration profile will convect downstream with a velocity a, and is described by the linear advection equation. In practice, molecular diffusion and turbulence will cause the concentration profile to change. With the simple one-dimensional model we cannot model turbulence, however the effect of molecular diffusion can be included by determining the diffusive flux. This flux is described by Fourier's Law of heat conduction (the diffusion of a chemical concentration is similar to diffusion of heat):

$$\text{diffusive flux} = -D\frac{\partial \rho}{\partial x}.$$

Combining this with the advective flux, $a\rho$, we obtain the **advection-diffusion equation**:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(a\rho - D\frac{\partial \rho}{\partial x}\right) = 0$$

Note that for the advection-diffusion equation, the flux function now depends on $\frac{\partial \rho}{\partial x}$ as well as ρ. The advection-diffusion equation is a *parabolic* equation, while the linear advection equation is *hyperbolic*. This means that the advection-diffusion equation always has smooth solutions, even if the initial data is discontinuous, while the linear advection equation admits discontinuities. We will consider some solutions of the linear advection equation later in the lecture.

1.3.2 Inviscid Burgers' Equation

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Flux function $f(u) = \frac{1}{2}u^2$

Conservation law :

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial \frac{1}{2}u^2}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

N3

Note 3 **Burgers' Equation**

The actual equation studied by Burgers includes a viscous term:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}.$$

This is one of the simplest models that includes the nonlinear and viscous effects of fluid dynamics. Again, when we include the viscous term, the equation becomes parabolic and does not admit discontinuous solutions.
An important aspect of the flux function, that will be used later, is that it is convex; i.e. $f''(u) = \frac{1}{2} > 0$.

1.3.3 Traffic Flow

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Let $\rho(x,t)$ denote the density of cars (vehicles/km) and $u(x,t)$ the velocity. Since cars are conserved,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

Assume that u is a function of ρ:

$$u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

where $0 \le \rho \le \rho_{\max}$ and $u_{\max}$ is some maximum speed (the speed limit?). N4

Note 4 ***Traffic Flow Problem***

Typically on a highway, we wish to drive at some speed $u_{\max}$, but in heavy traffic we slow down. At some point, the highway reaches its maximum capacity of cars, $\rho_{\max}$, and our velocity is zero. The simplest model for this relationship between velocity and density is that given above. This function has been found to provide a fairly good model for actual traffic flows. For example, for the Lincoln tunnel a good fit to actual data was obtained using the function

$$f(\rho) = a\rho\log\left(\frac{\rho_{\max}}{\rho}\right),$$

which has a similar shape to our linear relation (see [W]).
We point out that with either of the two relationships between car density and velocity, the flux is a concave function of ρ; i.e. $f''(\rho) \le 0$.

1.3.4 Buckley-Leverett Equation

SLIDE 9

Consider a two phase (oil and water) fluid flow in porous medium. Let $0 \le u(x,t) \le 1$ represent the saturation of water.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

This equation has applications in oil reservoir simulation where one models the flow of oil and water through porous rock or sand. So u varies between 0 and 1 : $u = 0$ represents a flow of pure oil, $u = 1$ represents pure water. N5

$$f(u) = \frac{u^2}{u^2 + a\,(1-u)^2}$$

a: constant ~ 1

Note 5 **The Buckley-Leverett Equations**

For the most part, we consider equations where $f(u)$ is convex (or a concave) function of the unknown variable. In the convex (or concave) case, the solution of an initial discontinuos data distribution (Riemann problem) is always either a shock or a rarefaction (or expansion) wave. When f is not convex (nor concave), the solution might involve both. The Buckley-Leverett equation is a simple example where this situation may occur.

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2 Smooth Solutions

2.1 Total Derivative

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Recall the primitive form of the conservation law

$$\frac{\partial u}{\partial t} + a(u)\,\frac{\partial u}{\partial x} = 0$$

The total time variation of $u(x,t)$, on an arbitrary curve $x = x(t)$, in the $x-t$ plane, is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\,\frac{\partial u}{\partial x}$$

2.2 Characteristics

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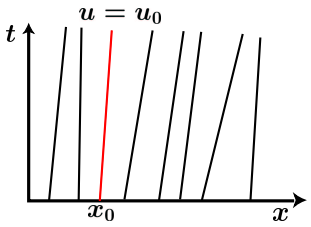
$$\textbf{If}\quad \frac{dx}{dt} = a(u) \quad \Rightarrow \quad \frac{du}{dt} = 0 \quad \Rightarrow \quad u = u_0 \text{ (constant)}$$

The curves $x = x(t)$, such that $\dfrac{dx}{dt} = a(u)$ are called **characteristics**

u constant $\Rightarrow$ $a(u)$ constant $\Rightarrow$
characteristics are straight lines

The characteristics are straight lines in the $x-t$ plane along which u is constant. If $u(x_0, 0) = u_0$, the characteristic passing through $x = x_0$, $t = 0$, is the solution of the following initial value problem $dx/dt = a(u_0)$, $x(0) = x_0$; i.e. $x = x_0 + a(u_0)t$.



$$\frac{dx}{dt} = a(u_0) \quad \Rightarrow \quad x = x_0 + a(u_0)\, t$$ N6

Note 6 **Characteristics**

The slope of the characteristic lines is determined by the initial condition $u_0(x)$, except for the trivial case in which f is a linear function of u. In this latter case, the slope characteristics is constant i.e. the characteristics are parallel.
We note that, for our problems, the characteristics are straight lines, even in the non-linear case, because f is determined by u only. For systems of equations, or for scalar equations with either a source term or a flux function that depends explicitly on x, the characteristics are no longer straight lines.
If we solve a problem on a finite domain, the number of boundary conditions to be prescribed, in the non-linear case, depends on the data itself. That is, in those boundaries with incoming characteristics a boundary condition will be required. Similarly, the solution at those boundaries with outgoing characteristics will be determined by the interior. We can see therefore that in the 1D case we can require, two, one or no boundary condition.
For nonlinear conservation laws and arbitrary data, the characteristics may cross within finite time. This would suggest a multi-valued solution which does not make any sense physically. We will see that just at the point where the characteristics start crossing, the solution becomes discontinuous. At this point, the differential primitive form of the equation, on which we are basing our solution procedure is no longer valid.

2.3 Examples

2.3.1 Linear Advection Equation

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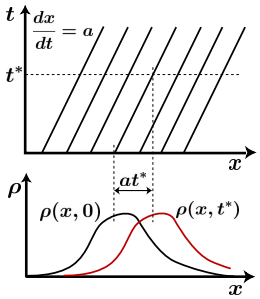

Solution

$$\rho(x,t) = \rho_0(x - at)$$

Characteristic lines

$$x = x_0 + at$$

2.3.2 Burgers' Equation

SLIDE 14

Recall $f(u) = \frac{1}{2}u^2$, so $a(u) = u$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Solution : $u(x,t) = u_0(x - ut)$

The solution is constant along the characteristic lines defined by $x - ut = x_0$.

We note that the above solution is defined implicitly (e.g. the definition of the function requires the function itself) and therefore it is often not very useful. We can verify however by direct differentiation that it is in fact a solution of the partial differential equation; i.e.,

$$u_x = u_0' - u_0' u_x t \quad \Rightarrow \quad u_x = \frac{u_0'}{1 + u_0' t}$$

$$u_t = -u_0' u - u_0' u_t t \quad \Rightarrow \quad u_t = \frac{-u_0' u}{1 + u_0' t}.$$

SLIDE 15

Consider the initial data

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

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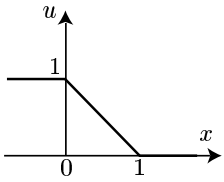

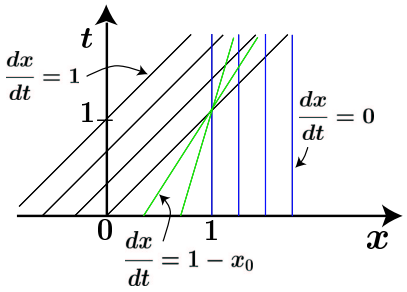

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For $t \leq 1$

For $x \leq t$: $\quad \dfrac{dx}{dt} = 1 \;\rightarrow\; x = t + x_0 \;\rightarrow \boxed{u(x,t) = 1}$

For $t < x < 1$: $\quad \dfrac{dx}{dt} = 1 - x_0 \;\rightarrow\; x = (1 - x_0)t + x_0$

$$\boxed{u(x,t) = 1 - x_0 = \frac{1-x}{1-t}}$$

For $x \geq 1$: $\quad \dfrac{dx}{dt} = 0 \;\rightarrow\; x = x_0 \;\rightarrow\; \boxed{u(x,t) = 0}$

For $t < x < 1$, we first solve for the characteristic lines. In this case they are defined by $\frac{dx}{dt} = u$. Since u is constant along each characteristic, we know that on each line $\frac{dx}{dt} = u_0$. Once we have determined the characteristic lines, we use the fact that u is constant along each line to determine the overall solution.

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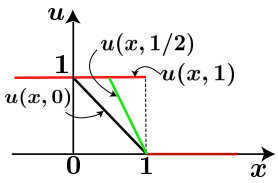

For $t = 1$

$$u(x,t) = \begin{cases} 1 & x < 1 \\ 0 & x > 1 \end{cases}$$

At $t = 1$ the solution develops a discontinuity. This corresponds to the time at which the characteristics first cross.

The procedure breaks down for $t > 1$

3 Discontinuous Solution

3.1 Shock Formation

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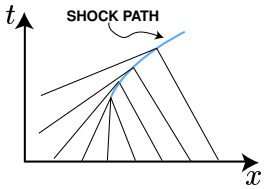
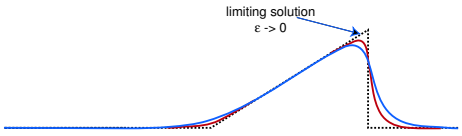


When the characteristics cross, the function $u(x,t)$ has an infinite slope. A discontinuity or **shock** forms, and the differential equation is no longer valid. N7 E1

Note 7 ***Vanishing Viscosity Approach***

After the characteristics have crossed, there are some points x where more than one characteristic leads back to $t = 0$. This would imply that the solution is multi-valued at such a point, which in most cases is not physically realisable. The correct physical behaviour can be determined by recalling that the inviscid Burgers' equation was a simplified version of a viscous equation with a term $\epsilon \frac{\partial^2 u}{\partial x^2}$ on the righthand side. If the initial data is smooth and ϵ is very small, then this term is negligible compared to the lefthand-side terms, and the solution of the viscous equation is almost identical to that of the inviscid equation. However, as the discontinuity begins to form, $\frac{\partial^2 u}{\partial x^2}$ becomes very large, and the viscous term becomes important. This term keeps the solution smooth for all time (recall the equation is now parabolic), and determines the correct physical nature of the system as shown in the figure below.



This behaviour is evident in the equations governing fluid flow. The Euler equations, which ignore the viscous terms, are hyperbolic and admit discontinuous solutions. Conversely, the Navier-Stokes equations are parabolic, and the viscosity ensures that the solution is always smooth.

▷ ***Exercise 1*** (from [LV]) Show that the viscous Burgers' equation has a travelling wave solution of the form $u^\epsilon(x,t) = w(x - st)$ by deriving an ODE for w and verifying that this ODE has solutions of the form

$$w(y) = u_R + \frac{1}{2}(u_R - u_L)[1 - \tanh((u_R - u_L)y/4\epsilon)]$$

with $s = (u_R + u_L)/2$. Note that $w(y) \to u_L$ as $y \to -\infty$ and $w(y) \to u_R$ as $y \to +\infty$. Sketch this solution and indicate how it varies as $\epsilon \to 0$. ■

3.2 The Riemann Problem

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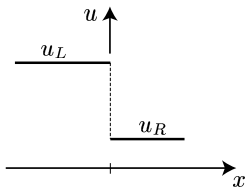
In order to understand the behaviour of the solution at discontinuities it is useful to start with a simplified problem. The Riemann problem is a conservation law together with piecewise constant data having a single discontinuity.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$



3.3 Shock Path

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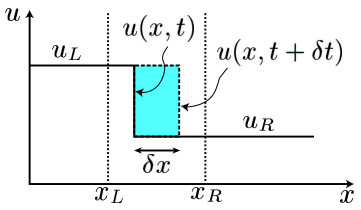
To determine the shock path we need to go back to the ***integral form*** *of the conservation law which is still valid across the shock.*

$$\frac{\partial}{\partial t} \int_{x_L}^{x_R} u \, dx = -\left(f(u_R) - f(u_L)\right)$$



The boundaries x_R and x_L are taken sufficiently close to the shock, so that spatial variations of the solution away from the shock become unimportant. They are also taken sufficiently far apart from the shock so that the boundary will not interfere with the shock motion over a time interval δt.

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$$-\frac{1}{\delta t}(u_R - u_L)\delta x = -\left(f(u_R) - f(u_L)\right)$$

Shock speed

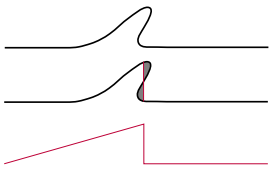
$$\boxed{s = \frac{\delta x}{\delta t} = \frac{f(u_R) - f(u_L)}{u_R - u_L} \equiv \frac{[f]}{[u]}}$$

Rankine-Hugoniot jump condition

N8

Note 8 **The Equal Area Rule**

One technique that is sometimes useful for determining the discontinuous solution is to start with the solution constructed using characteristics. This solution may be multi-valued if characteristics cross. The multi-valued parts can be eliminated by inserting shocks at appropriate locations. The shock location can be determined by the *equal area rule* which is depicted in the figure below. The shock is placed so that the shaded areas cut off on either side have equal areas.



3.3.1 Example

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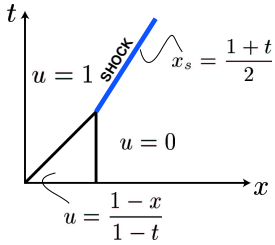
For our example,

$$s = \frac{[f]}{[u]} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}$$

$$x_s = \frac{1+t}{2}$$



3.3.2 Variable States

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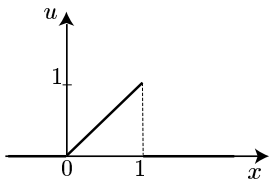
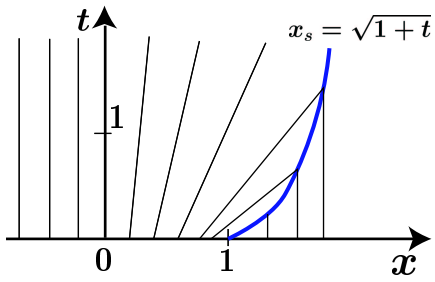
If u is not constant at both sides of the shock, the jump condition still applies **locally** $\Rightarrow$ shock path is **curved**.

Example:
Burgers' equation with

$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$





The characteristics emanating from $(0,1)$ *at* $t = 0$ *are given by*

$$x = x_0(1+t)$$

on which

$$u(x,t) = x_0 = \frac{x}{1+t}$$

provided characteristics have not intersected. To the right of shock the characteristics are vertical with $u = 0$ *on each.*
So at the shock, x_s*,*

$$\begin{aligned} \frac{dx_s}{dt} = \frac{f(u_R) - f(u_L)}{u_R - u_L} &= \frac{1}{2}\frac{u_R^2 - u_L^2}{u_R - u_L} \\ &= \frac{1}{2}(u_R + u_L) \\ &= \frac{1}{2}(0 + \frac{x_s}{1+t}) \\ &= \frac{1}{2}\frac{x_s}{1+t}. \end{aligned}$$

This ODE has solution

$$x_s = \sqrt{1+t},$$

which is the shock position separating u_R *and* $u_L = \frac{1}{\sqrt{1+t}}$.

3.3.3 Manipulating the Conservation Law

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Consider

$$\frac{\partial u}{\partial t} + \frac{\partial \frac{1}{2}u^2}{\partial x} = 0 \qquad (\boldsymbol{A})$$

Multiply by u

$$u\frac{\partial u}{\partial t} + u\frac{\partial \frac{1}{2}u^2}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \frac{1}{2}u^2}{\partial t} + \frac{\partial \frac{1}{3}u^3}{\partial x} = 0$$

Let $v = \frac{1}{2}u^2$, then we can write

$$\frac{\partial v}{\partial t} + \frac{\partial \frac{\sqrt{8}}{3}\, v^{\frac{3}{2}}}{\partial x} = 0 \qquad (\boldsymbol{B})$$

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Characteristics:

$\boldsymbol{A}$ $\quad \frac{dx}{dt} = a(u) = u$

$\boldsymbol{B}$ $\quad \frac{dx}{dt} = a(v) = \frac{\sqrt{8}}{3}\,\frac{3}{2}\sqrt{v} = \sqrt{2v} = u$

Shock speed:

$\boldsymbol{A}$ $\quad s_A = \frac{1}{2}\left(u_R + u_L\right)$

$\boldsymbol{B}$ $\quad s_B = \frac{2}{3}\,\frac{u_R^3 - u_L^3}{u_R^2 - u_L^2} \neq \boldsymbol{s_A}$

N9

Use conservation law for physically conserved quantity

Note 9 **Shock speeds**

Discontinuous solutions depend on the quantity being conserved. In the original equation, we conserve u, while in the second equation we conserve $\frac{1}{2}u^2$. Recall the shock speeds are given by $s = \frac{[f]}{[u]}$. For the original equation,

$$s_1 = \frac{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2}{u_R - u_L} = \frac{1}{2}\left(u_R + u_L\right).$$

For the modified equation,

$$s_2 = \frac{\frac{1}{3}u_R^3 - \frac{1}{3}u_L^3}{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2} = \frac{2}{3}\frac{u_R^3 - u_L^3}{u_R^2 - u_L^2} \neq \frac{1}{2}\left(u_R + u_L\right).$$

It is therefore very important to always use the conserved quantity to calculate shock speed.

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4 Weak Solutions

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A natural way to define a generalized solution of the inviscid equation that does not require differentiability is to go back to the integral form of the conservation law, and say that $u(x,t)$ is a generalized solution of the equation if the integral form of the equation is satisfied for all x_L and x_R.

An alternative to the above approach is to write a weak statement in a form which requires less smoothness from the solution.

Multiply $u_t + f_x = 0$ by $\phi(x,t) \in \mathcal{C}_0^1(\mathbb{R} \times \mathbb{R}_+)$

$\mathcal{C}_0^1$ is the space of continuous functions with continuous first derivative that have compact support; i.e. that are zero at infinity.

$$\int_0^\infty \int_{-\infty}^\infty \phi(u_t + f_x)\, dxdt = 0$$

Integrating by parts

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f]\, dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)dx = 0$$

The above statement will be modified accordingly for a bounded spatial domain, by incorporating the appropriate boundary terms.

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If above statement is satisfied for all $\phi \in \mathcal{C}_0^1(\mathbb{R} \times \mathbb{R}_+)$ then $u(x,t)$ is a **weak solution**

Weak solutions are essentially solutions that satisfy the differential equation where the solution is smooth, and the jump condition at discontinuities. All (strong) solutions of the differential equation are also weak solutions; the reverse is obviously not true. Unfortunately, there is a price to pay by enlarging the class of solutions.

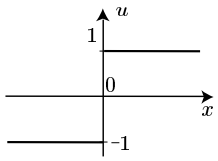
Weak solutions to conservation laws **are** often **non unique**.

4.1 Non-uniqueness

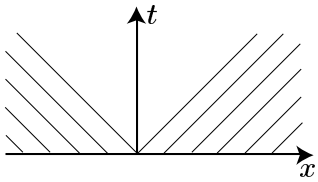
4.1.1 Example : Burgers' equation

SLIDE 30

$$u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$



SLIDE 31

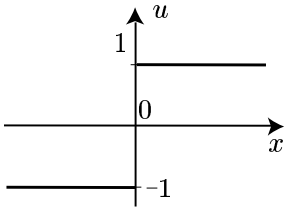


The initial condition is in fact a solution for all times

Shock wave (Solution **A**)

$$s = \frac{u_L + u_R}{2} = 0$$

$$u(x,t) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

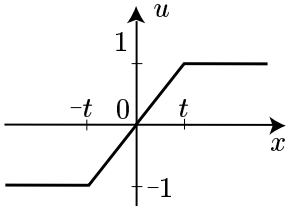


SLIDE 32

Another possible solution is

Rarefaction wave (Solution **B**)
(sometimes called expansion waves)

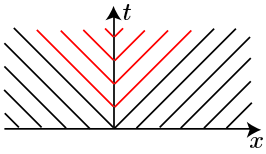
$$u(x,t) = \begin{cases} -1 & x < -t \\ \frac{x}{t} & -t \le x \le t \\ 1 & x > t \end{cases}$$



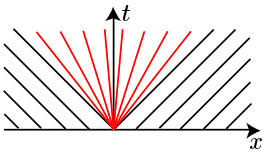
In fact, it is possible to construct infinitely many weak solutions. Try it!

SLIDE 33

Solution **A**



Solution **B**



4.2 Entropy Condition

SLIDE 34

Which is the physically relevant solution?

Criterion : the physical solution satisfies

$$\lim_{\epsilon \to 0} \left(\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} \right)$$

We consider the more general viscous solution, and take the limit as $\epsilon \to 0$. This is the vanishing viscosity solution discussed earlier. This condition is not particularly easy to work with. We will examine below some equivalent forms [L,S,LV] of expressing this entropy condition.

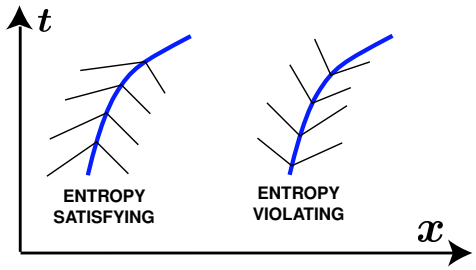
4.2.1 Convex (concave) fluxes

SLIDE 35

When $f(u)$ is convex i.e. $f''(u) \geq 0$ (or concave i.e. $f''(u) \leq 0$) for all u, the entropy condition can be written as

$$\boxed{a(u_L) = f'(u_L) > s > f'(u_R) = a(u_R)}$$

Characteristics must run **into** the shock for increasing t, not emerge from it. SLIDE 36



4.2.2 Example

SLIDE 37

Solution **A**

$a(u_L) = u_L = -1$
$s = \frac{1}{2}(u_R + u_L) = 0$
$a(u_R) = u_R = 1$

Entropy condition is violated - characteristics emerge from the shock

Solution **B** No shock $\Rightarrow$ OK

4.2.3 Oleinik's Condition

SLIDE 38

Applicable to general flux functions
$u(x,t)$ is the entropy satisfying solution if all discontinuities satisfy the property that

$$\boxed{\frac{f(u) - f(u_L)}{u - u_L} \geq s \geq \frac{f(u) - f(u_R)}{u - u_R}}$$

for all u between u_L and u_R.

4.2.4 Entropy Functions

Now we consider yet another characterization of the entropy conditions which is very general and can be readily extended to systems of equations.

SLIDE 39

$U(u)$ is an **entropy function** if it is **positive**, **convex**, and there exists a corresponding **entropy flux** such that

$$F'(u) = U'(u)f'(u)$$

For smooth solutions

$$\Rightarrow \quad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Proof: Multiply the original equation $u_t + f_x = 0$ by U' and the above results follows.

SLIDE 40

The function $u(x,t)$ is the entropy satisfying solution of the governing equations if, for all convex entropy functions $U(u)$ and corresponding entropy fluxes $F(u)$, the inequality

$$\boxed{\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0}$$

is satisfied in the weak sense. N10

If $f(u)$ is **convex** we only need to check for **one** $U(u)$

Note 10 ***Entropy Functions***

This is an alternative approach to enforcing the entropy condition. In gas dynamics, a physical quantity exists called the *entropy* that is known to be constant along particle paths in smooth flow, and to jump to a higher value as the gas crosses a shock. The entropy can never jump to a lower value (Second Law of Thermodynamics) and this gives the extra condition which enables us to select the physically correct solution.
The above inequality is derived by multiplying the viscous equation by $U'(u)$

$$\begin{aligned} U'u_t + U'f &= \epsilon U'u_{xx} \\ U_t + F_x &= \epsilon U'u_{xx} \\ U_t + F_x &= \epsilon(U_{xx} - U''u_x^2) \end{aligned}$$

since U is convex, $U'' > 0$ and so

$$U_t + F_x \leq \epsilon U_{xx}$$

and letting $\epsilon \to 0$ gives,

$$U_t + F_x \leq 0$$

in a weak sense.
The weak form of the entropy inequality is thus

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t U + \phi_x F]\, dxdt + \int_{-\infty}^\infty \phi(x,0)dx \leq 0$$

to be satisfied for all $\phi \in \mathcal{C}_0^1(\mathbb{R} \times \mathbb{R}_+)$ with $\phi(x,t) \geq 0$ for all x and t.
Note that becasue we have now an inequality, we must require that the weighting functions ϕ be positive.

4.2.5 Example

SLIDE 41

Burgers' equation $f(u) = \frac{1}{2}u^2$

Take $U(u) = u^2$ and $F(u) = \frac{2}{3}u^3$
Entropy inequality

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x \leq 0$$

5 Application Examples

5.1 Traffic Flow

SLIDE 42

Governing equation : $\rho_t + f(\rho)_x = 0$

$$f(\rho) = \rho u = \rho\, u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

$$f'(\rho) = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right), \quad f''(\rho) = -2\frac{u_{\max}}{\rho_{\max}} < 0$$

The flux is a concave function of the unknown ρ.

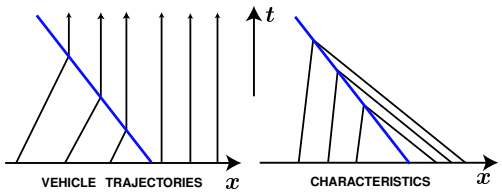
$$s = \frac{[f]}{[\rho]} = u_{\max}\left(1 - \frac{\rho_L + \rho_R}{\rho_{\max}}\right)$$

5.1.1 "Traffic Jam"

SLIDE 43

Consider $\rho_R = \rho_{\max}$ and $\rho_L < \rho_{\max}$ $\Rightarrow$ **Shock**

Models moving cars encountering a traffic jam and instantaneously stopping.



$s < 0$ and the shock propagates to the left N11

Note 11 ***Traffic Flow : shock solution***

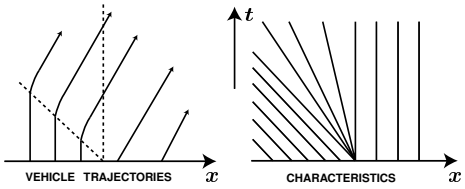
The cars are moving at speed $u_L > 0$ when they encounter the traffic jam. When they slam on their brakes, they instantaneously reduce their velocity to zero, and the density jumps from ρ_L to $\rho_{\max}$ (this is the shock). The shock location moves to the left as more cars join the traffic jam ($s < 0$). We can verify that characteristics run into the shock and that the entropy condition is thus satisfied.

5.1.2 "Green Light"

SLIDE 44

Consider $0 < \rho_R < \rho_L < \rho_{\max}$ $\Rightarrow$ **Rarefaction**.

This might model the startup of cars after a light turns green.



$\rho_L = \rho_{\max}$, $\rho_R = \frac{1}{2}\rho_{\max}$ N12

Note 11 ***Traffic flow: rarefaction solution***

The cars to the left initially have zero velocity (they are stopped at the intersection). They begin to accelerate as the cars in front of them move. Since the velocity is inversely proportional to the density, each car can only speed up

by allowing the distance between cars to increase. Therefore we see a gradual acceleration and spreading out of cars (the rarefaction wave).
Note also there is another weak solution to this problem. This solution contains a shock and corresponds to drivers accelerating instantaneously from $u_L = 0$ to $u_R > 0$ as the cars in front move out of the way. This solution is not physically viable in terms of the original problem. In terms of mathematical viability, although it satisfies the governing equations, it is inadmissible since the characteristics run out of the shock and therefore the entropy condition is violated.

5.1.3 "Sound Speed"

SLIDE 45

For smooth solutions information travels with speed $f'(\rho)$

$$\rho_t + f'(\rho)\rho_x = 0$$

Consider a nearly constant solution

$$\rho(x,t) = \rho_0 + \varepsilon\rho_1(x,t)$$

If ε is small we can model $\rho_1(x,t)$ with the linear equation

$$\rho_{1t} + f'(\rho_0)\rho_{1x} = 0$$

The initial data simply propagates unchanged with velocity $f'(\rho_0)$.

SLIDE 46

$$f'(\rho_0) = u_{\max}\left(1 - \frac{2\rho_0}{\rho_{\max}}\right), \quad u_0 = u_{\max}\left(1 - \frac{\rho_0}{\rho_{\max}}\right)$$

$f'(\rho_0) \le u_0$ cars travel ahead of disturbances

But ...

$\rho_0 < \frac{1}{2}\rho_{\max}$	$f'(\rho_0) > 0$	disturbances move forward
$\rho_0 > \frac{1}{2}\rho_{\max}$	$f'(\rho_0) < 0$	disturbances move backward

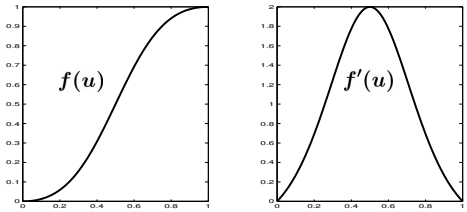
$\rho_0 = \frac{1}{2}\rho_{\max}$: "sonic point"

5.2 Buckley-Leverett Equation

SLIDE 47

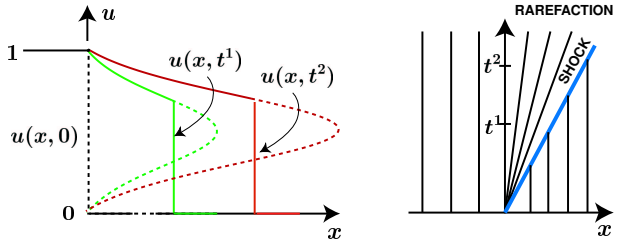
$$f(u) = \frac{u^2}{u^2 + a(1-u)^2}, \quad a = 1$$

Here we consider the Riemann problem consisting of a discontinuos initial data $u = 1$ for $x < 0$ and $u = 0$ for $x > 0$.



We see that the flux function in this case is neither convex nor concave. In fact, it has an inflexion point.

SLIDE 48



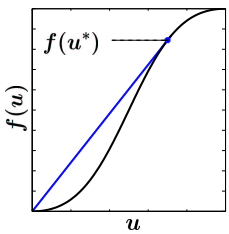
N13

Physically we see that as the water moves in, it displaces a certain fraction of the oil immediately. Behind the shock there is a mixture of oil and water with less and less oil as times goes on.

We note that for non-convex flux functions it is possible to have a Riemann solution which involves both a shock and a rarefaction wave. If the flux function had more inflexion points, the solution might involve more shocks and rarefaction waves.

Note 13 **Graphic determination of the shcok location**

The solution to the Riemann problem can be determined from the graph of $f(u)$ in a simple manner.



The point of tangency u^* is precisely the post-shock value. The straight line represents the shock jumping from $u = 0$ to $u = u^*$ and the segment of $f(u)$ above the tangency point represents the rarefaction wave. Note that the slope of the straight segment is precisely the shock speed. The fact that the line is tangent to the curve means that the shock moves at the same speed as the characteristic at the edge of the rarefaction fan.
It can bee seen that if the shock were connected to some point below u^*, then the solution would be multi-valued, and if the shock were connected to some point above u^* the entropy condition would be violated.

---

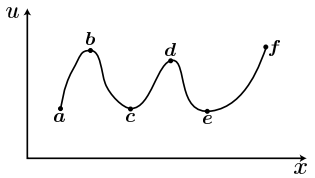
6 Total Variation

6.1 Definition

SLIDE 49

We examine now a property of the scalar conservation laws that will be useful in developing numerical approximations. The total variation of a function u is defined as:

$$\boxed{TV(u) = \int \left| \frac{\partial u}{\partial x} \right| dx}$$



$$TV(u) = |u_b - u_a| + |u_c - u_b| + |u_d - u_c| + |u_e - u_d| + |u_f - u_e|$$

We note that the total variation can be related to the maxima and minima of the function as follows:

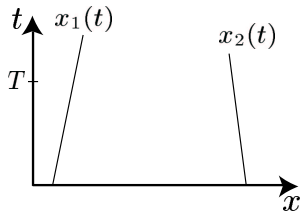
$$\begin{aligned} TV(u) &= |u_b - u_a| + |u_c - u_b| + |u_d - u_c| + |u_e - u_d| + |u_f - u_e| \\ &= 2(u_b + u_d) - 2(u_c + u_e) - u_a + u_f \\ &= 2\left(\sum maxima - \sum minima\right) \end{aligned}$$

6.2 Continuous Case

SLIDE 50

Let the above function u correspond to the initial condition for our conservation law.

Consider the total variation of $u(x,t)$ between two points $x_1(t)$ and $x_2(t)$ lying on two characteristics.



SLIDE 51

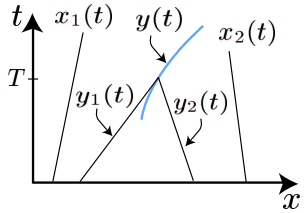
If there are **no shocks** between $x_1(t)$ and $x_2(t)$, then the extrema will not change, and the **total variation will stay constant with time**.

$$\boxed{TV(u(x,t)) = TV(u(x,0))}$$

6.3 Discontinuous Case

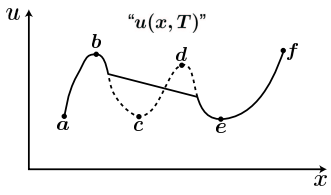
SLIDE 52

The solution at a shock is determined by $y_1(t)$ and $y_2(t)$ provided the shock is entropy satisfying.



SLIDE 53

The function in this figure is not exactly $u(x,T)$, but has the same total variation of $u(x,T)$ since it has the same extrema.



$$TV(u(x,T)) \quad = \quad |u_b - u_a| + |u_c - u_e| + |u_f - u_e|$$

By inspection we see that

$$\begin{aligned} TV(u(x,T)) &= |u_b - u_a| + |u_b - u_e| + |u_f - u_e| \\ &\leq |u_b - u_a| + |u_c - u_b| + |u_d - u_c| + |u_e - u_d| + |u_f - u_e| \\ &= TV(u(x,0)) \end{aligned}$$

If there are shocks: $TV(u(x,T)) \leq TV(u(x,0))$
In general,

$$\boxed{\frac{d}{dt}\left(\int \left|\frac{\partial u}{\partial x}\right| dx\right) \leq 0}$$

The total variation of an entropy satisfying solution is a non-increasing function of time.

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