Partial Differential Equations: An Overview

Lecture 1

1 Model Equation

1.1 Convection-Diffusion

SLIDE 1

$$\boxed{\frac{\partial u}{\partial t} + \boldsymbol{U} \cdot \nabla u = \kappa \nabla^2 u + f}$$

N1

$$\nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\boldsymbol{U}$, $\kappa > 0$, f, given functions of (x, y)

Scalar, Linear, Parabolic equation N2

Despite its apparent simplicity, this equation appears in a wide range of disciplines ranging from Heat Transfer to Financial Engineering. During this course we will make extensive use of this equation, and several of the limiting cases contained therein, to illustrate the numerical techniques that will be presented.

In some cases $\boldsymbol{U}$, k, and f will be functions of the solution u, in which case the equation is said to be nonlinear.

Note 1 Derivation of the Convection–Diffusion Equation for Heat Transfer

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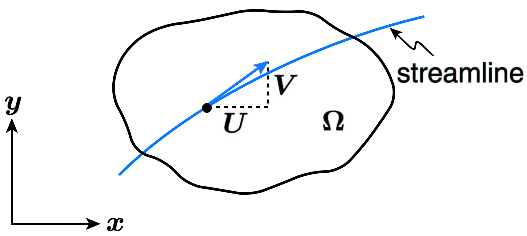
We sketch below the derivation of the Convection–Diffusion equation for the particular problem of Heat Transfer in a moving fluid. Consider a velocity field $\boldsymbol{U} = (U(x, y), V(x, y))$ which is (for simplicity) time independent and incompressible, e.g.

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$



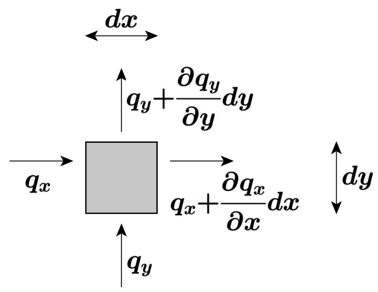
A streamline is a curve which is obtained by integrating the vector field and corresponds to the trajectory of a fluid parcel

$$\left.\frac{dy}{dx}\right|_{\text{streamline}} = \frac{V}{U} \ .$$

For a given parcel we can write the equation expressing the balance of energy as

$$\left.\frac{dE}{dt}\right|_{\text{fixed parcel}} = \underbrace{Q_c}_{\substack{\text{rate of net heat}\\ \text{transferred}\\ \text{into parcel}}} + \underbrace{f^*}_{\substack{\text{rate of volumetric}\\ \text{heat generation}\\ \text{inside parcel}}}$$

where $E = \rho c T$ is the internal energy per unit volume, T is the temperature, ρ is the density and c is the specific heat. The term Q_c can be expressed in terms of the heat flux $\boldsymbol{q} = (q_x, q_y)$, by considering an infinitesimal parcel of size $dx\,dy$.



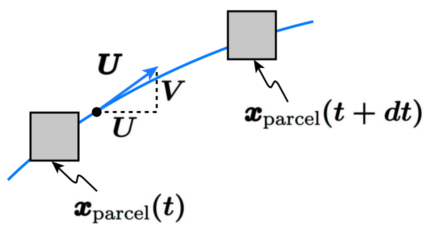
The net rate of the heat transferred into the parcel, per unit area, will thus be

$$Q_c = \frac{-1}{dx\,dy}\left[\frac{\partial q_x}{\partial x}\,dx\,dy + \frac{\partial q_y}{\partial y}\,dx\,dy\right] = -\nabla \cdot \boldsymbol{q} \ .$$

The heat flux is finally related to the temperature through Fourier's law

$$\boldsymbol{q} = -k \nabla T$$

where k is the (constant) thermal conductivity of the fluid.
The derivative term $\frac{dE}{dt}$ is called a material derivative because it is associated with a fixed fluid parcel that moves with the flow.



Expressing the material derivative (Lagrangian notion) in terms of fixed time and space derivatives (Eulerian notion) we obtain

$$\left.\frac{dE}{dt}\right|_{\text{fixed parcel}} = \rho\, c\, \frac{d}{dt} T(\boldsymbol{x}_{\text{parcel}}(t), t)$$

$$= \rho\, c \left(\frac{\partial T}{\partial x} \underbrace{\frac{dx_{\text{parcel}}}{dt}}_{U} + \frac{\partial T}{\partial y} \underbrace{\frac{dy_{\text{parcel}}}{dt}}_{V} + \frac{\partial T}{\partial t} \right)$$

$$= \rho\, c \left(\frac{\partial T}{\partial t} + U\, \frac{\partial T}{\partial x} + V\, \frac{\partial T}{\partial y} \right)$$

$$= \rho\, c \left(\frac{\partial T}{\partial t} + \boldsymbol{U} \cdot \nabla T \right)$$

which then yields the final form of the convection-diffusion equation after dividing by ρc and defining $\kappa = \frac{k}{\rho\, c}$, $f = \frac{f^*}{\rho\, c}$.

Note 2 ***Classification of PDE's (Optional Reading)***

We review the classification of first and second order linear partial differential equations in two independent variables. This classification is useful, not only to identify solution methods applicable to a particular equation type, but also to determine the character of the solutions. For two independent variables all equations can be classified as shown below. We note that for equations with more than two independent variables, the classification is far more complex as some equations may become of mixed type (see [F] for additional reading)

Within this note, (x, y), denotes the independent variables and $\phi(x, y)$, the dependent variable, or solution of the PDE.

First Order PDE's

First order partial differential equations are always of **hyperbolic** type. A general linear first order equation can be written as

$$A\phi_x + B\phi_y = F(x, y, \phi)$$

where A and B may be functions of x and y, but not of ϕ. If we write $d\phi = \phi_x dx + \phi_y dy$, then

$$Ad\phi + \phi_y(Bdx - Ady) = Fdx \ .$$

Along the lines (characteristics) such that $Bdx - Ady = 0$,

$$A\frac{d\phi}{dx} = F \qquad \text{(ODE)}.$$

Characteristics are $Bx - Ay = \psi$, for any ψ, and the general solution becomes

$$\phi = \frac{1}{A}\int F\ dx + g(\psi) = \frac{1}{A}\int F\ dx + g(Bx - Ay).$$

Where g is an arbitrary function to be determined by the initial and boundary conditions.

Second Order PDE's
A linear second order partial differential equation can be written as

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$$

where A, B and C may be functions of x and y. Based on the local value of the coefficients the equations are classified as follows:

$B^2 - 4AC > 0$	**Hyperbolic**
$B^2 - 4AC = 0$	**Parabolic**
$B^2 - 4AC < 0$	**Elliptic**

Note that an equation may change type from one point to another since the coefficients may be functions of x and y. We will typically assume that, when we say that an equation is of a given type, it remains of the same type over the whole domain.
Consider a *valid* change of independent variables $\zeta = \zeta(x, y)$, $\eta = \eta(x, y)$, such that

$$J = \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix}, \qquad |J| \neq 0.$$

Then,

$$\begin{aligned} \phi_x &= \phi_\zeta \ \zeta_x + \phi_\eta \ \eta_x \\ \phi_{xx} &= \phi_{\zeta\zeta} \ \zeta_x^2 + 2\phi_{\zeta\eta} \ \zeta_x \eta_x + \phi_{\eta\eta} \ \eta_x^2 + \phi_\zeta \ \zeta_{xx} + \phi_\eta \ \eta_{xx} \\ &\vdots \end{aligned}$$

The transformed equation becomes

$$a\phi_{\zeta\zeta} + b\phi_{\zeta\eta} + c\phi_{\eta\eta} = f(\zeta, \eta, \phi, \phi_\zeta, \phi_\eta)$$

with

$$\begin{aligned} a &= A \ \zeta_x^2 + B \ \zeta_x \zeta_y + C \ \zeta_y^2 \\ b &= 2A \ \zeta_x \eta_x + B \ (\zeta_x \eta_y + \zeta_y \eta_x) + 2C \ \zeta_y \eta_y \\ c &= A \ \eta_x^2 + B \ \eta_x \eta_y + C \ \eta_y^2 \end{aligned}$$

THEOREM: This classification is invariant under valid non-singular transformations.
Proof: From above $b^2 - 4ac = (B^2 - 4AC) \ (\zeta_x \eta_y - \zeta_y \eta_x)^2 = (B^2 - 4AC) \ |J|^2$.

CANONICAL FORMS

HYPERBOLIC case ($B^2 - 4AC > 0$):
In this case it is always possible to choose ζ, η so that $a = c = 0$, i.e.

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0, \qquad A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0,$$

$$\zeta_x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}\zeta_y, \qquad \eta_x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}\eta_y,$$

Then, the equation becomes

$$\boxed{\phi_{\zeta\eta} = F'(\zeta, \eta, \phi, \phi_\zeta, \phi_\eta).}$$

An alternative form can be obtained by setting $X = \zeta + \eta,\ \ Y = \zeta - \eta$:

$$\boxed{\phi_{XX} - \phi_{YY} = F''(X, Y, \phi, \phi_X, \phi_Y)}$$

PARABOLIC case $(B^2 - 4AC = 0)$:
Here, we can only set a (or c) to zero (not both), otherwise ζ and η are not independent. If we set $a = 0$, then

$$\frac{\zeta_x}{\zeta_y} = -\frac{B}{2A}\ .$$

It can be verified, by direct evaluation, that in this case $b = 0$, in which case we can pick η to be *any* function such that $|J| \neq 0$, and the equation becomes:

$$\boxed{\phi_{\eta\eta} = F'(\zeta, \eta, \phi, \phi_\zeta, \phi_\eta).}$$

ELLIPTIC case $(B^2 - 4AC < 0)$:
This case is identical to the hyperbolic case but now ζ and η are complex conjugates $(B^2 - 4AC < 0)$. Take $X = \zeta + \eta,\ \ Y = i(\zeta - \eta)$ and the equation becomes:

$$\boxed{\phi_{XX} + \phi_{YY} = F'(X, Y, \phi, \phi_X, \phi_Y).}$$

1.1.1 Applications

SLIDE 2

$$\boxed{\frac{\partial u}{\partial t} + \boldsymbol{U} \cdot \nabla u = \kappa \nabla^2 u + f}$$

If u is ...

- Temperature → **Heat Transfer**
- Pollutant Concentration → **Coastal Engineering**
- Probability Distribution → **Statistical Mechanics**

  This equation is known in Statistical Mechanics as the Fokker–Planck equation.

- Price of an Option → **Financial Engineering**

  This equation is known in Financial Engineering as the Black–Scholes equation.

- ...

In some of the above cases the equation is slightly different (e.g. particular non–constant coefficients), however the basic form remains invariant.

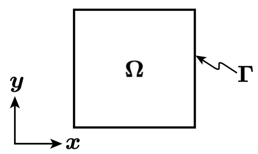
2 Limiting Cases

2.1 Elliptic Equations

SLIDE 3

Poisson Equation

$$-\kappa \nabla^2 u = f \quad \text{in } \Omega$$



Convection-Diffusion

$$\boldsymbol{U} \cdot \nabla u = \kappa \nabla^2 u \quad \text{in } \Omega$$

- "Smooth" solutions

  — *even when the boundary conditions or f are not smooth.*

- The domain of dependence of $u(x, y)$ is Ω

This means that a small perturbation of f, or boundary conditions, anywhere in the domain will alter the value of $u(x, y)$.

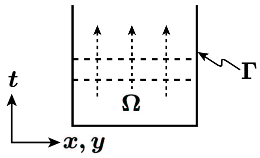
The solution of elliptic equations will be studied extensively in this course. For these equations we will be presenting solution techniques using Finite Differences, Finite Elements and Boundary Integral Methods.

2.2 Parabolic Equations

SLIDE 4

Heat Equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + f \quad \text{in } \Omega$$



- "Smooth" solutions

  — *even when the initial, boundary conditions, or f, are not smooth.*

- The domain of dependence of $u(x, y, T)$ is $(x, y, t < T)$

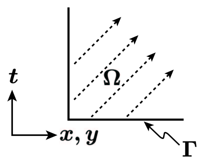
In this course we will address the solution of parabolic equations using Finite Difference and Finite Element Methods.

2.3 Hyperbolic Equations

SLIDE 5

Wave Equation (First order)

$$\frac{\partial u}{\partial t} + \boldsymbol{U} \cdot \nabla u = f \quad \text{in } \Omega$$

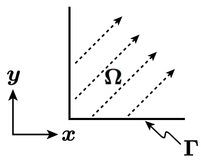


- Non-smooth solutions
- Characteristics : $\frac{d\boldsymbol{x_c}}{dt} = \boldsymbol{U}(\boldsymbol{x_c}(t))$
- The domain of dependence of $u(\boldsymbol{x}, T)$ is $(\boldsymbol{x_c}(t), t < T)$

SLIDE 6

Convection Equation

$$\boldsymbol{U} \cdot \nabla u = f \quad \text{in } \Omega$$



- Non-smooth solutions
- Characteristics are streamlines of $\boldsymbol{U}$, e.g. $\frac{d\boldsymbol{x_c}}{ds} = \boldsymbol{U}$
- The domain of dependence of $u(\boldsymbol{x})$ is $(\boldsymbol{x_c}(s), s < 0)$

We will present Finite Difference and Finite Volume Methods for solving hyperbolic equations. In particular, Finite Volume Methods will be extended to deal with non-linear hyperbolic equations.

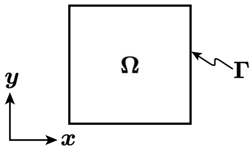
2.4 Eigenvalue Problem

SLIDE 7

Find non-trivial pairs (u, λ)

$$\kappa \nabla^2 u + \lambda u = 0 \quad \text{in} \quad \Omega$$

with **homogeneous** conditions on Γ



From the mathematical classification point of view, the eigenvalue equation is a semi-linear elliptic equation.

- Non-linear
- "Closely" related to other problems

We shall see below that the eigenvalue problem of a given spatial operator is closely related to the temporal evolution of the solution of the associated time-dependent equation. The particular eigenvalue problem shown here is closely related to the Heat Equation.

2.5 One Spatial Variable

In some cases we will consider the above equations involving only one spatial variable.

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Unknown		Equation
$u(x)$	:	$-u_{xx} = f$
$u(x)$	:	$U u_x = \kappa u_{xx}$
$u(x,t)$	:	$u_t = \kappa u_{xx}$
$u(x,t)$	:	$u_t + U u_x = 0$
$(u(x), \lambda)$	:	$u_{xx} + \lambda u = 0$

3 Fourier Analysis

3.1 Definition

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Let $g(x)$ be an "arbitrary" periodic real function with period 2π N3

$$\boxed{g(x) = \sum_{k=-\infty}^{\infty} g_k \; e^{ikx} \quad (k \text{ integer}) \quad .}$$

The following orthogonality relationship satisfied by the Fourier modes can be easily verified

$$\int_0^{2\pi} e^{ikx} e^{-ik'x} \; dx = 2\pi \; \delta_{kk'} \quad \text{(orthogonality)}$$

We recall that $\delta_{kk'}$ *is the Kronocker symbol and is equal to* 1 *if* $k = k'$ *and* 0 *otherwise. Using the above relationship, the coefficients* g_k *can be computed directly as*

$$g_k = \frac{1}{2\pi} \int_0^{2\pi} g(x) \; e^{-ikx} \; dx$$

Note that when $g(x)$ *is real,* $g_k = g^*_{-k}$*, where* * *denotes complex conjugate.*

Note 3 **Fourier series**

Fourier series are only defined for functions which satisfy the following regularity condition

$$\int_0^{2\pi} |g(x)|^2 \; dx < \infty.$$

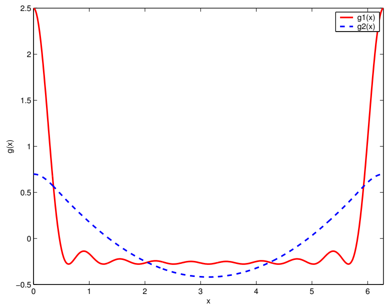
We also note that the decomposition presented above can be written in an equivalent form as

$$g(x) = \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx,$$

with $a_k = g_k + g_{-k}$ and $b_k = i(g_k - g_{-k})$.

3.2 Example

SLIDE 10



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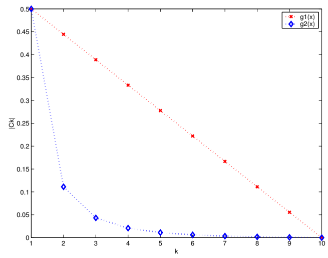

Rate at which $|g_k| \to 0$ for $|k|$ large determines **smoothness**

The above figures show two functions together with a plot of the amplitude of their Fourier coefficients. An important feature of the Fourier coefficients is that they are directly associated with the smoothness of the original function: we can intuitively see that the less oscillatory the function, the faster $|g_k|$ will tend to zero as $|k|$ becomes large, since we will need less "high frequency" (small wavelength) contributions.

3.3 Differentiation

SLIDE 12

$$\boxed{u(x) = \sum_{k=-\infty}^{\infty} u_k\, e^{ikx} \quad \textbf{or} \quad u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t)\, e^{ikx}}$$

$$\frac{\partial^n u}{\partial x^n} = \sum_{k=-\infty}^{\infty} (ik)^n\, u_k\, e^{ikx} \qquad\qquad \frac{\partial u}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt}\, e^{ikx}$$

$$\begin{array}{lllllll} n & = & 2m & \to & (ik)^n & = & (-1)^m\, k^{2m} \quad \text{(real)} \\ n & = & 2m-1 & \to & (ik)^n & = & -i(-1)^m\, k^{2m-1} \quad \text{(imaginary)} \end{array}$$

Since differentiation decreases the rate of decrease of the Fourier coefficients, we must assume that our function is sufficiently smooth ($|u_k| \rightarrow 0$ fast enough) in order to meaningfully perform this operation. In particular, it can be shown that, if a function has p bounded derivatives, $|u_k||k|^{p+1} < \infty$ as $|k| \rightarrow \infty$.

3.4 Poisson Equation

SLIDE 13

$$\boxed{-u_{xx} = f \qquad x \in (0, 2\pi)}$$

with

$$\begin{aligned} u(0) &= u(2\pi), \\ u_x(0) &= u_x(2\pi), \end{aligned}$$

The number and type of boundary conditions required for this equation will be discussed further during the course; however it should be clear that, in one dimension, a linear equation involving an n-th order spatial operator will require n boundary conditions.

and

$$\int_0^{2\pi} u\, dx = 0, \qquad \int_0^{2\pi} f\, dx = 0$$

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Note 4 **Existence and Uniqueness of the Solution**

Physically, there can be no net generation since the flux in ($u_x(0)$) equals the flux out ($u_x(2\pi)$). Also note that, from the equation u is free to "float" and the condition $\int_0^{2\pi} u\, dx = 0$ pins the solution. Physically, the level of u is determined by initial conditions for the heat equation of which $-u_{xx} = f$ is the long–time limit.

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$$u = \sum_{k=-\infty}^{\infty} u_k\, e^{ikx} \;, \quad f = \sum_{k=-\infty}^{\infty} f_k\, e^{ikx} \qquad (f_0 = 0)$$

$$-u_{xx} = \sum_{k=-\infty}^{\infty} k^2\, u_k\, e^{ikx} \quad \rightarrow \quad \boxed{u_k = \frac{f_k}{k^2}} \quad (u_0 = 0)$$

$\Rightarrow$ – the solution u is **smoother** than f

3.4.1 Example

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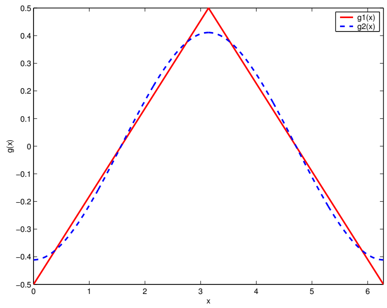

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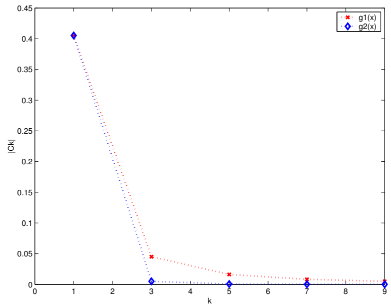

3.5 Heat Equation

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$$\boxed{u_t = \kappa u_{xx} \qquad x \in (0, 2\pi)}$$

with

$$\begin{aligned} u(0,t) &= u(2\pi, t), \\ u_x(0,t) &= u_x(2\pi, t), \\ u(x,0) &= u^0(x) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{ikx} \end{aligned}$$

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$$u = \sum_{k=-\infty}^{\infty} u_k(t) \, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt} \, e^{ikx} \ , \qquad u_{xx} = \sum_{k=-\infty}^{\infty} -k^2 \, u_k \, e^{ikx}$$

$$\frac{du_k}{dt} = -\kappa k^2 \, u_k$$

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$$\frac{du_k}{dt} = -\kappa k^2 u_k \ , \quad u_k(t=0) = u_k^0 \ , \quad \Rightarrow \quad u_k(t) = u_k^0 \, e^{-\kappa k^2 t}$$

$$\boxed{u(x,t) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{-\kappa k^2 t} \, e^{ikx}}$$

$\Rightarrow$ – exponential decay of initial condition (**dissipation**)
– higher decay for "higher modes" (larger k) $\equiv$ **smoothness**

3.5.1 Example

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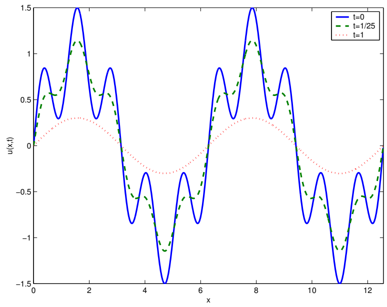

3.6 Wave Equation

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$$\boxed{u_t + U u_x = 0 \qquad x \in (0, 2\pi)}$$

with

$$\begin{aligned} u(0,t) &= u(2\pi, t), \\ u(x,0) &= u^0(x) = \sum_{k=-\infty}^{\infty} u_k^0 \, e^{ikx} \end{aligned}$$

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$$u = \sum_{k=-\infty}^{\infty} u_k(t) \, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt} \, e^{ikx} \ , \qquad u_x = \sum_{k=-\infty}^{\infty} iku_k \, e^{ikx}$$

$$\frac{du_k}{dt} = -iUk\,u_k$$

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$$\frac{du_k}{dt} = -iUku_k \ , \quad u_k(0) = u_k^0 \quad \Rightarrow \quad u_k(t) = u_k^0\, e^{-iUkt}$$

$$\boxed{u(x,t) = \sum_{k=-\infty}^{\infty} u_k^0\, e^{-iUkt}\, e^{ikx} = \sum_{k=-\infty}^{\infty} u_k^0\, e^{ik(x-Ut)} = u^0(x-Ut)}$$

$\Rightarrow$
- no decay, **propagation** with wave speed $c = U$
- no **dispersion** (c constant) $\equiv$ invariant shape

3.6.1 Example

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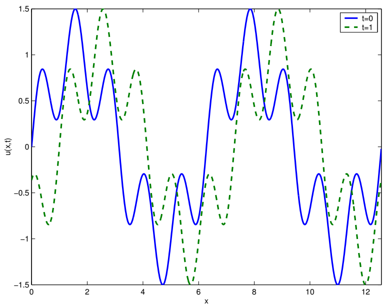

3.7 General Operator

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$$\boxed{u_t = \frac{\partial^n u}{\partial x^n} \qquad x \in (0, 2\pi)}$$

with

$$\begin{aligned} u(0,t) &= u(2\pi,t), \\ u_x(0,t) &= u_x(2\pi,t), \\ &\vdots \\ u_x^{(n-1)}(0,t) &= u_x^{(n-1)}(2\pi,t), \\ u(x,0) &= u^0(x) \end{aligned}$$

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$$u = \sum_{k=-\infty}^{\infty} u_k(t)\, e^{ikx}$$

$$u_t = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt}\, e^{ikx} \ , \qquad u_x^{(n)} = \sum_{k=-\infty}^{\infty} (ik)^n u_k\, e^{ikx}$$

$$\boxed{\frac{du_k}{dt} = \sigma\, u_k}$$ $$\boxed{\sigma = (ik)^n}$$ SLIDE 27

n	σ	Feature
1	ik	**Propagation**, $c = -\sigma/ik = -1$ (no Dispersion)
2	$-k^2$	**Decay**
3	$-ik^3$	**Propagation**, $c = +k^2$ (and **Dispersion**)
4	k^4	**Growth** ($-u_{xxxx}$ much faster Decay than u_{xx})
.	.	

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Note 5 ***Dispersion***

Whenever the speed of propagation depends on the wavelength $2\pi/k$ (or wavenumber k), we say that the wave is dispersive. A consequence of dispersion is that the shape of the solution will change as the wave propagates. We see from the above table that for $n = 1$ (the wave equation), the speed of propagation is constant for all modes and therefore any initial shape will be preserved as it propagates. For $n = 3$, $c = -k^2$ and waves with high $|k|$ will propagate a lot faster than the waves with small $|k|$, and therefore any initial condition will change its shape as the solution evolves.
We also note that when different derivative terms are present, the solution will exhibit a behaviour which is the result of combining the different terms present. For instance an equation such as $u_t = -u_x + u_{xx}$ will exhibit both propagation and decay; this can easily be seen by noting that in this case $\sigma = -ik - k^2$.

3.8 Eigenvalue Problem

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$$\boxed{u_{xx} + \lambda u = 0 \qquad x \in (0, 2\pi)}$$

with

$$\begin{aligned} u(0) &= u(2\pi), \\ u_x(0) &= u_x(2\pi) \end{aligned}$$

Need to determine non-trivial pairs $(u^n(x), \lambda^n)$

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It can be easily verified that the eigenvalues are:

$$\lambda^n = n^2, \quad \text{for} \quad n = 1, 2, \ldots$$

The eigenvectors associated with λ^n are:

$$u_1^n(x) = e^{inx}, \quad u_2^n(x) = e^{-inx}, \quad \text{for} \quad n = 1, 2, \ldots$$

Eigenmodes ≡ Fourier modes

We note that the two eigenvectors corresponding to the eigenvalue λ^n can be combined into $\sin nx$ and $\cos nx$. We note also that in addition to the eigenvalues and eigenvectors shown above, $\lambda^0 = 0$, $u^0(x) = 1$ is also an eigenvector.

Since the eigenmodes coincide with the Fourier modes, we can regard our original Fourier expansion as an expansion in terms of eigenmodes. In fact, this latter interpretion is quite general and applies in situations in which the Fourier modes either do not exist, or do not coincide with the eigenmodes of the differential operator.

4 Eigenvalue Expansions

4.1 Formal Extension

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$$\frac{\partial u}{\partial t} = \mathcal{L}u$$

with homogeneous boundary conditions

N6

$$u(x, y, t) = \sum_{n=0}^{\infty} a_n(t) u^n(x, y)$$

$$(u^n, \lambda^n) \quad \text{solution of} \quad \mathcal{L}u - \lambda u = 0$$

Here we are assuming that the eigenvalues/eigenvectors have been ordered in such a way that if an eigenvalue has a multiplicity m it will contribute m terms to the sumation.

We observe that if we attempt the solution of the problem $u_t = \mathcal{L}u$, using the method of separation of variables

$$u(x, y, t) = X(x, y)\ T(t)$$

then

$$\frac{T'(t)}{T(t)} = \frac{\mathcal{L}X(x, y)}{X(x, y)} = -r$$

where r is the separation variable. The function $X(x, y)$ will then be required to solve the eigenvalue problem $\mathcal{L}X + rX = 0$, and the eigenvalue will be precisely the separation variable.

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$$\mathcal{L}u = \sum_{n=0}^{\infty} \lambda^n a_n u^n, \qquad \frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} \frac{da_n}{dt}\ u^n$$

$$\frac{da_n}{dt} = \lambda^n a_n \quad \Rightarrow \quad a_n(t) = a_n^0\ e^{\lambda^n t}$$

$$\boxed{u(x, y, t) = \sum_{n=0}^{\infty} a_n^0\ e^{\lambda^n t}\ u^n(x, y)}$$

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Eigenvalues determine temporal evolution of the associated time-dependent problem.

Higher λ
$\Updownarrow$
Higher **decay/frequency**
$\Updownarrow$
More Oscillations

Higher spatial oscillations can be expected since a larger eigenvalue will generate larger time variations that will have to be matched, through the original equation, with higher spatial derivatives.

For the heat equation solved on arbitrary domains we will expect the eigenvalues to be real and negative (decay). On the other hand, for the wave equation the eigenvalues will be purely imaginary (propagation). Finally, we will expect the eigenvalues to be general complex numbers when convective and diffusive terms are present in the equation.

REFERENCES

[F] S.J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover 1993.