Finite Difference Discretization of Elliptic Equations: 1D Problem

Lectures 2 and 3

Model Problem

Poisson Equation in 1D

Boundary Value Problem (BVP)

$$\boxed{-u_{xx}(x) = f(x)}$$ N1

$$x \in (0,1), \quad u(0) = u(1) = 0, \quad f \in \mathcal{C}^0$$ N2

N3

Describes many simple physical phenomena (e.g.):

- Deformation of an elastic bar N4
- Deformation of a string under tension N5
- Temperature distribution in a bar N6

Model Problem

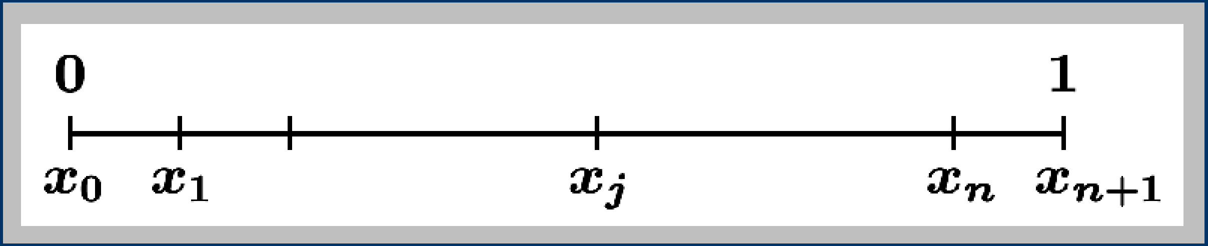
Poisson Equation in 1D

Solution Properties

- The solution $u(x)$ always **exists**
- $u(x)$ is always "**smoother**" than the data $f(x)$
- If $f(x) \geq 0$ for all x, then $u(x) \geq 0$ for all x
- $||u||_\infty \leq (1/8)||f||_\infty$ N7
- Given $f(x)$ the solution $u(x)$ is **unique** N8

Subdivide interval $(0,1)$ into $n+1$ equal subintervals

$$\Delta x = \frac{1}{n+1}$$



$$x_j = j\Delta x, \qquad \hat{u}_j \approx u_j \equiv u(x_j)$$

for $\quad 0 \leq j \leq n+1$

For example . . .

$$v''(x_j) \approx \frac{1}{\Delta x}(v'(x_{j+1/2}) - v'(x_{j-1/2}))$$

$$\approx \frac{1}{\Delta x}(\frac{v_{j+1} - v_j}{\Delta x} - \frac{v_j - v_{j-1}}{\Delta x})$$

$$= \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2}$$

for Δx small

Numerical Solution

Finite Differences

Equations...

$-u_{xx} = f$ suggests . . .

$$-\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{\Delta x^2} = f(x_j) \quad 1 \leq j \leq n$$

$$\hat{u}_0 = \hat{u}_{n+1} = 0$$

$$\Longrightarrow \qquad \boxed{A\,\underline{\hat{u}} = \underline{f}}$$

Numerical Solution

Finite Differences

...Equations

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad \underline{\hat{u}} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-1} \\ \hat{u}_n \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{pmatrix}$$

(Symmetric)

$$A \in \mathbb{R}^{n \times n} \qquad \underline{\hat{u}},\ \underline{f} \in \mathbb{R}^n$$

Is A non-singular ?

For any $\underline{v} = \{v_1, v_2, \ldots, v_n\}^T$

$$\underline{v}^T \, A \, \underline{v} = \frac{1}{\Delta x^2}\left(v_1^2 + \sum_{i=2}^{n}(v_i - v_{i-1})^2 + v_n^2\right)$$

Hence $\boxed{\underline{v}^T \, A \, \underline{v} > 0, \text{ for any } \underline{v} \neq 0}$ (A is **SPD**) **N9**

$A \, \underline{\hat{u}} = \underline{f}$: $\underline{\hat{u}}$ **exists** and is **unique** **N10**

Numerical Solution

Finite Differences

Example...

$$-u_{xx} = (3x + x^2)e^x, \qquad x \in (0,1)$$

with

$$u(0) = u(1) = 0.$$

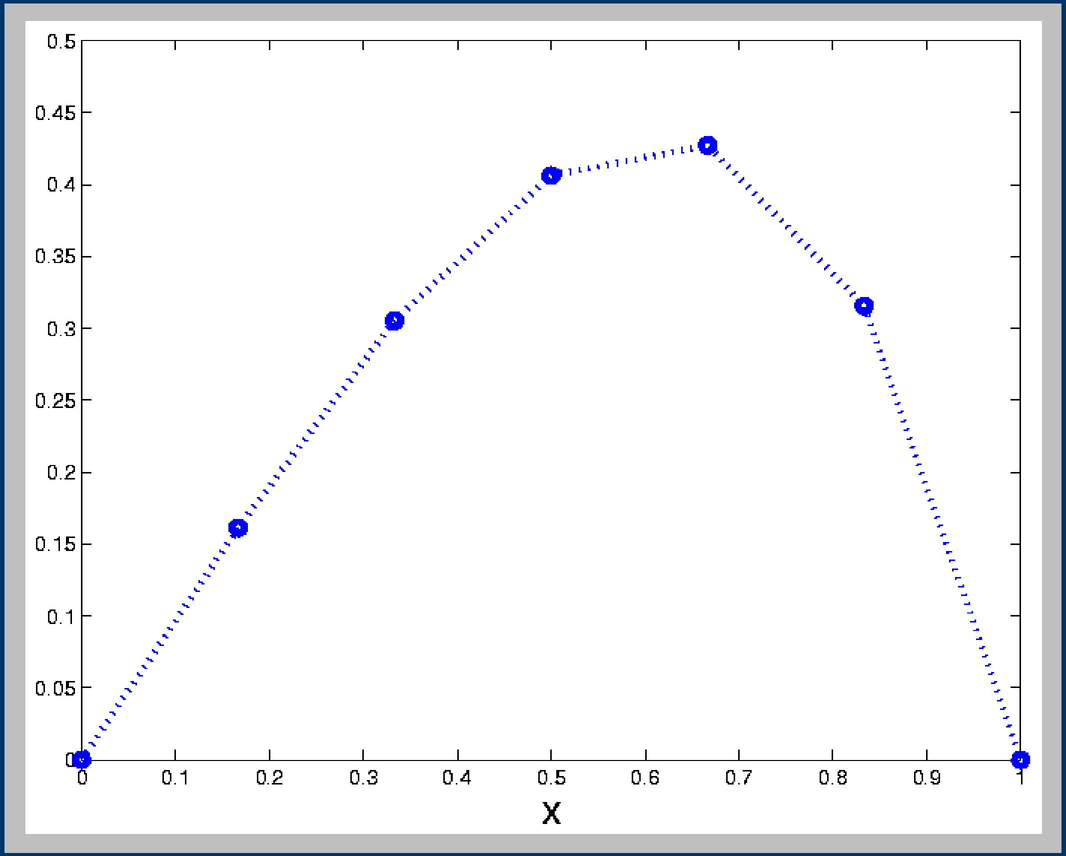
Take $n = 5, \;\; \Delta x = 1/6 \;\; \ldots$

Numerical Solution

Finite Differences

...Example

$\hat{u}$



Convergence ?

1. Does the discrete solution $\hat{u}$ retain the qualitative propeties of the continuous solution $u(x)$?

2. Does the solution become more accurate when $\Delta x \to 0$?

3. Can we make $|u(x_j) - \hat{u}_j|$ for $0 \leq j \leq n+1$ arbitrarily small?

Discretization Error Analysis

Properties of A^{-1}

Let

$$A^{-1} = \{\alpha_{ij}\}_{1 \le i,j \le n}$$

- Non-negativity **N11**

$$\alpha_{ij} \ge 0, \qquad \text{for} \qquad 1 \le i, j \le n$$

- Boundedness **N12**

$$0 \le \sum_{j=1}^{N} \alpha_{ij} \le \frac{1}{8}, \qquad \text{for} \qquad 1 \le i \le n$$

Discretization Error Analysis

Qualitative Properties of $\hat{u}$

$$\underline{f} \geq 0 \;\rightarrow\; \hat{\underline{u}} \geq 0$$

$$\underline{\hat{u}} = A^{-1}\, \underline{f}$$

If

$$f_j = f(x_j) \geq 0\,, \qquad \text{for} \quad 1 \leq j \leq n$$

Then

$$\hat{u}_i = \sum_j \alpha_{ij} f_j \geq 0\,, \qquad \text{for} \quad 1 \leq i \leq n$$

Discretization Error Analysis

Qualitative Properties of $\hat{u}$

Discrete Stability

$$\underline{\hat{u}} = A^{-1}\,\underline{f}$$

$$||\underline{\hat{u}}||_\infty = \max_i |\hat{u}_i| = \max_i \left(\left|\sum_j \alpha_{ij} f_j\right|\right)$$

$$\leq \max_i \left(\sum_j \alpha_{ij}\right) \max_i |f_i|$$

$$\leq \frac{1}{8} ||\underline{f}||_\infty$$

Discretization Error Analysis

Truncation Error

For any $v \in C^4$ we can show that

N13

$$\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{\Delta x^2} = v''(x_j) + \frac{\Delta x^2}{12} v^{(4)}(x_j + \theta \Delta x)$$

$$-1 \leq \theta \leq 1$$

Take $u \equiv v \qquad (-u'' = f)$

$$-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{\Delta x^2} = f(x_j) \underbrace{- \frac{\Delta x^2}{12} u^{(4)}(x_j + \theta_j \Delta x)}_{\tau_j}$$

Discretization Error Analysis

Error Equation

Let $\boxed{e_j = u(x_j) - \hat{u}_j}$ be the **discretization error**.

$$-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{\Delta x^2} = f(x_j) + \tau_j$$

$$-\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{\Delta x^2} = f(x_j)$$

Subtracting

$$-\frac{e_{j+1} - 2e_j + e_{j-1}}{\Delta x^2} = \tau_j, \qquad 1 \leq j \leq n$$

and

$$e_0 = e_{n+1} = 0$$

Discretization Error Analysis

Error Equation

$$\boxed{A\,\underline{e} = \underline{\tau}}$$

$$\underline{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ e_N \end{pmatrix}, \qquad \underline{\tau} = \frac{\Delta x^2}{12} \begin{pmatrix} u^{(4)}(x_1 + \theta_1 \Delta x) \\ u^{(4)}(x_2 + \theta_2 \Delta x) \\ \vdots \\ \vdots \\ u^{(4)}(x_N + \theta_N \Delta x) \end{pmatrix}$$

Discretization Error Analysis

Using the discrete stability estimate on $\boldsymbol{A}\, \underline{\boldsymbol{e}} = \underline{\boldsymbol{\tau}}$

$$||\underline{e}||_\infty \leq \frac{1}{8}||\underline{\tau}||_\infty$$

or

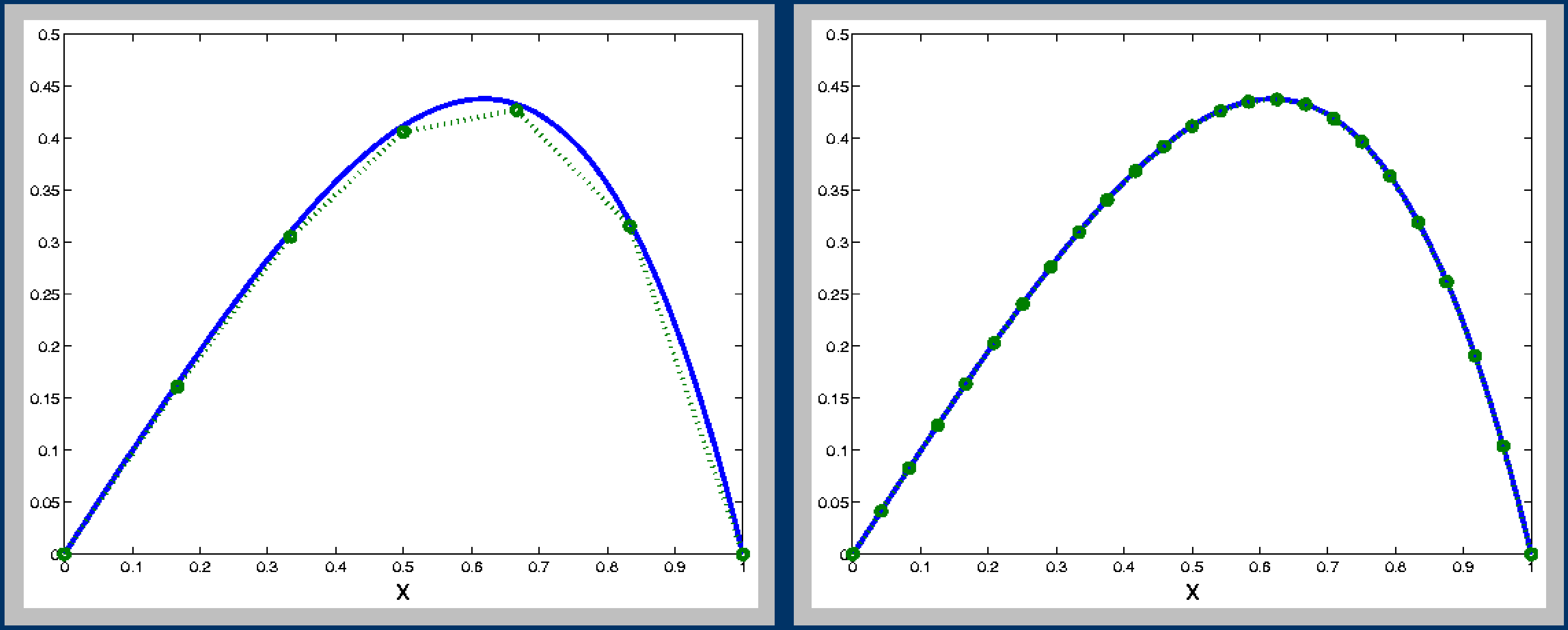
$$\max_{1 \leq i \leq n} |u(x_i) - \hat{u}_i| \leq \frac{\Delta x^2}{96} \max_{0 \leq x \leq 1} |u^{(4)}(x)|$$

A-priori Error Estimate

Discretization Error Analysis

Numerical Example

$$-u_{xx} = (3x + x^2)e^x, \quad x \in (0,1), \quad u(0) = u(1) = 0$$



$\Delta x = 1/6$ $\Delta x = 1/24$

Discretization Error Analysis

Numerical Example

EXAMPLE : $-u_{xx} = (3x + x^2)e^x, \;\; x \in (0, 1)$

$n+1$	$\|\|\underline{u} - \hat{\underline{u}}\|\|_\infty$
3	0.0227
6	0.0059
12	0.0015
24	$3.756e-04$
48	$9.404e-05$
96	$2.350e-05$
192	$5.876e-06$

Asymptotically,

$$\|\underline{u} - \hat{\underline{u}}\|_\infty \approx C \Delta x^\alpha$$

$$C = 0.216623$$

$$\alpha = 2.000$$

Summary

- For a simple model problem we can produce numerical approximations of **arbitrary accuracy**.

- An **a-priori error estimate** gives the asymptotic dependence of the solution error on the discretization size Δx.

Generalizations

Definitions

Consider a linear elliptic **differential equation**

$$\boxed{\mathcal{L}\, u = f}$$

and a **difference scheme**

$$\boxed{\hat{\mathcal{L}}\, \underline{\hat{u}} = \underline{\hat{f}}}$$

Generalizations

The difference scheme is **consistent** with the differential equation if:

For **all** smooth functions v

$$(\hat{\mathcal{L}}\underline{v} - \underline{\hat{f}})_j - (\mathcal{L}v - f)_j \to 0, \quad \text{for } j = 1, \ldots, n$$

when $\Delta x \to 0$.

$(\hat{\mathcal{L}}\underline{v} - \underline{\hat{f}})_j - (\mathcal{L}v - f)_j = \mathcal{O}(\Delta x^p)$ for all j

$\Rightarrow$ p is **order of accuracy**

Generalizations

Truncation Error

$$(\hat{\mathcal{L}}\underline{u} - \underline{\hat{f}})_j - \underbrace{(\mathcal{L}u - f)_j}_{=0} = \tau_j, \quad \text{for } \; j = 1, \ldots, n$$

or,

$$\hat{\mathcal{L}}\underline{u} - \underline{\hat{f}} = \underline{\tau} \; .$$

The truncation error results from inserting the exact solution into the difference scheme.

Consistency $\Rightarrow \; ||\underline{\tau}||_\infty = \mathcal{O}(\Delta x^p)$

Generalizations

Error Equation

Original scheme

$$\hat{\mathcal{L}}\,\underline{\hat{u}} = \underline{\hat{f}}$$

Consistency

$$\hat{\mathcal{L}}\,\underline{u} = \underline{\hat{f}} + \underline{\tau}$$

The error $\underline{e} = \underline{u} - \underline{\hat{u}}$ satisfies

$$\boxed{\hat{\mathcal{L}}\underline{e} = \underline{\tau}\ .}$$

Generalizations

Matrix norm

$$||M||_\infty = \sup_{\underline{v} \in \mathbb{R}^n} \frac{||M\underline{v}||_\infty}{||\underline{v}||_\infty}$$

N14

The difference scheme is **stable** if

$$||\hat{\mathcal{L}}^{-1}||_\infty \leq C \quad \text{(independent of } \Delta x\text{)}$$

Generalizations

$$||M||_\infty = \sup_{||\underline{v}||_\infty = 1} ||M\underline{v}||_\infty$$

$$= \sup_{||\underline{v}||_\infty = 1} (\max_i |\sum_{j=1}^{n} m_{ij} v_j|)$$

$$= \max_i (\sup_{||\underline{v}||_\infty = 1} |\sum_{j=1}^{n} m_{ij} v_j|) \qquad v_j = \text{sign}(m_{ij})$$

$$= \max_i \sum_{j=1}^{n} |m_{ij}|$$ **(max row sum)**

Generalizations

Convergence

Error equation

$$\underline{e} = \hat{\mathcal{L}}^{-1}\,\underline{\tau}$$

Taking norms

$$||\underline{e}||_\infty = ||\hat{\mathcal{L}}^{-1}\,\underline{\tau}||_\infty$$

$$\leq ||\hat{\mathcal{L}}^{-1}||_\infty\,||\underline{\tau}||_\infty$$

$$\leq \underbrace{||\hat{\mathcal{L}}^{-1}||_\infty\,C}_{C_1}\;\Delta x^p = C_1\,\Delta x^p$$

Generalizations

Summary

Consistency + Stability ⇒ Convergence

Convergence		Stability		Consistency
$\|\|\underline{e}\|\|_\infty$	$\leq$	$\|\|\hat{\mathcal{L}}^{-1}\|\|_\infty$	$\cdot$	$\|\|\underline{\tau}\|\|_\infty$

The Eigenvalue Problem

Model Problem

Statement

Find nontrivial (u, λ) such that

$$-u_{xx} = \lambda u, \qquad x \in (0,1)$$

$$u(0) = u(1) = 0;$$

denote solutions (u^k, λ^k), $k = 1, 2, \ldots,$ with

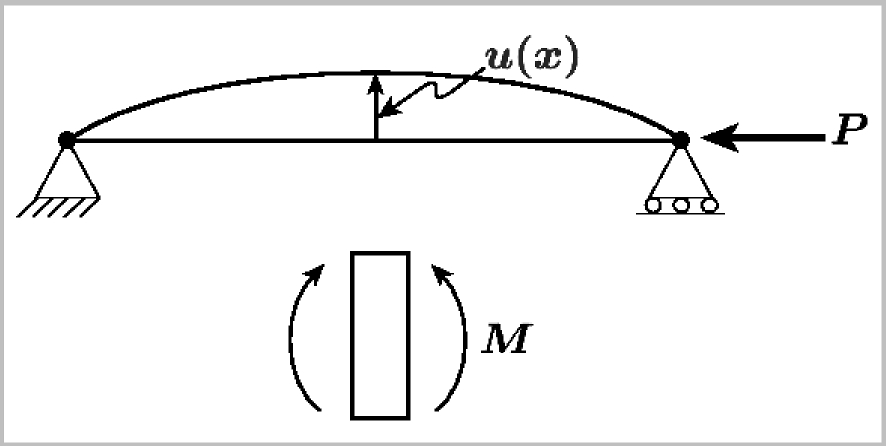
$$0 \leq \lambda^1 \leq \lambda^2 \leq \ldots$$

N15

The Eigenvalue Problem

Application

Axially Loaded Beam



- "Small" Deflection

$$EIu_{xx} = M_{internal}$$

- External Force

$$M_{external} = -Pu$$

Equilibrium $\Rightarrow u_{xx} + \frac{P}{EI}u = 0$

$$\lambda = P/EI$$

$$-u_{xx} = \lambda u, \qquad u(0) = u(1) = 0$$

The Eigenvalue Problem

Exact Solution

$$-u_{xx} - \lambda u = 0$$

$$\Downarrow$$

$$u = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$$

$$u(0) = 0 \Rightarrow B = 0$$

$u(1) = 0 \Rightarrow A = 0$ **or** $\lambda = k^2 \pi^2, k = 1, 2, \ldots$

The Eigenvalue Problem

Exact Solution

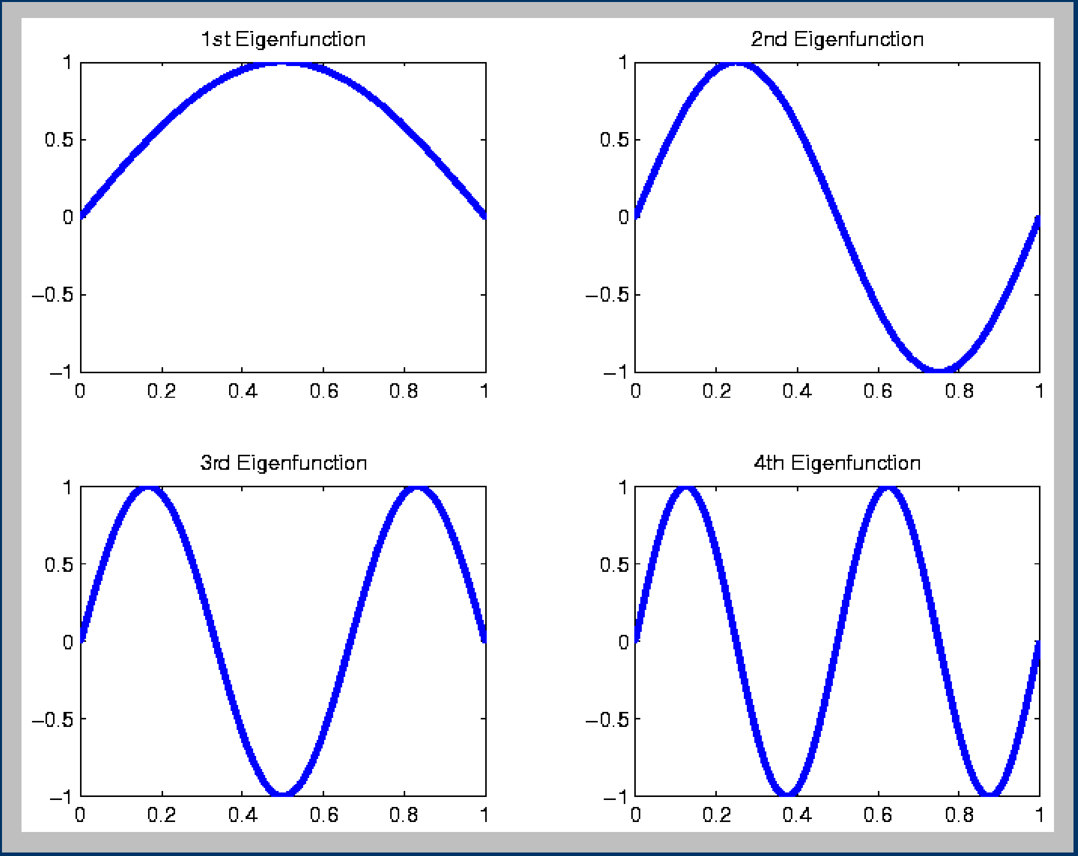
Thus (choose $A = 1$)

$$\left.\begin{array}{c} u^k = \sin k\pi x \\ \lambda^k = k^2\pi^2 \end{array}\right\} \quad k = 1, 2 \ldots$$

Larger $k \Rightarrow$ more oscillatory $u^k \Rightarrow$ larger λ.

The Eigenvalue Problem

Exact Solution



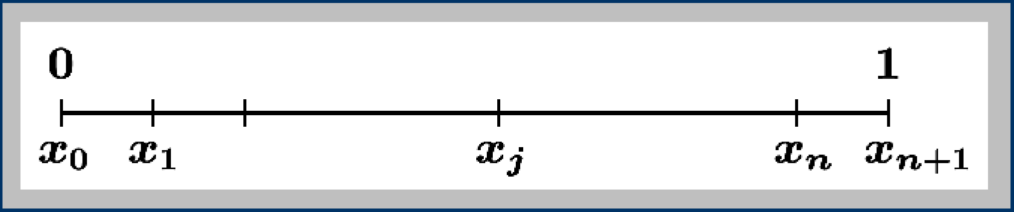
The Eigenvalue Problem

Discrete Equations

Difference Formulas

$$-u_{xx} = \lambda u, \qquad u(0) = u(1) = 0$$

$$\Downarrow$$



$$\Delta x = \frac{1}{n+1}$$

$$\frac{-1}{\Delta x^2}(\hat{u}_{j-1} - 2\hat{u}_j + \hat{u}_{j+1}) = \hat{\lambda}\hat{u}_j, \qquad j = 1, \ldots, n$$

$$\hat{u}_0 = \hat{u}_{n+1} = 0$$

The Eigenvalue Problem

Discrete Equations

Matrix Form

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \qquad \underline{\hat{u}} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{n-1} \\ \hat{u}_n \end{pmatrix}$$

$n \times n$ **SPD**

$$A\,\underline{\hat{u}} = \hat{\lambda}\,\underline{\hat{u}} \quad \Rightarrow \quad \underline{\hat{u}}^k, \hat{\lambda}^k, \quad k = 1, 2, \ldots, n$$

N17

N18

Claim that

$\underline{\hat{u}}^k \equiv \underline{u}^k$

$$\hat{u}_j^k = u^k(x_j) = \sin(k\pi x_j)$$

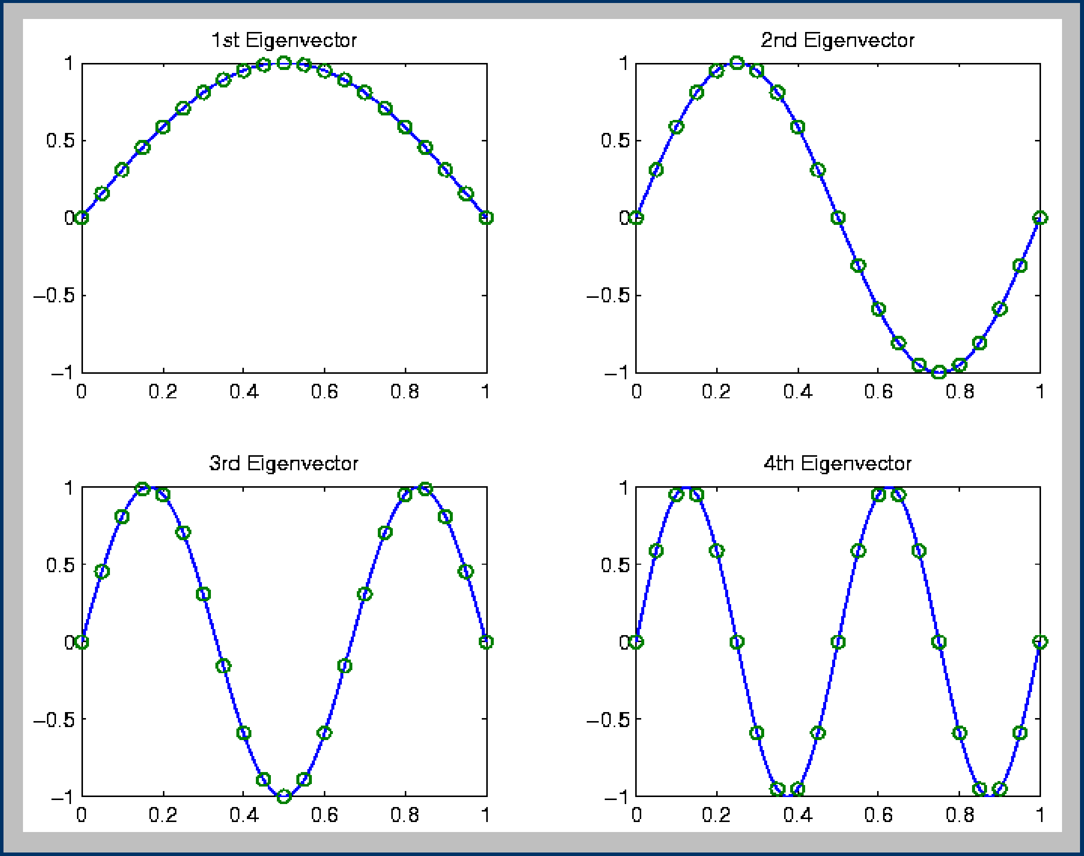
$$= \sin(k\pi j \Delta x) = \sin(\frac{k\pi j}{n+1}), \quad j = 1, \ldots, n$$

Note $\hat{u}_0^k = \hat{u}_{n+1}^k = 0$ since $\sin(0) = \sin(k\pi) = 0$.

The Eigenvalue Problem

Error Analysis

...Analytical Solution: $\underline{\hat{u}}^k, \hat{\lambda}^k$...



The Eigenvalue Problem

Error Analysis

...Analytical Solution: $\underline{\hat{u}}^k, \hat{\lambda}^k$...

What are $\hat{\lambda}^k$?

$$-\frac{1}{\Delta x^2}\{\hat{u}^k_{j-1} - 2\hat{u}^k_j + \hat{u}^k_{j+1}\}$$

$$= -\frac{1}{\Delta x^2}\{\sin(k\pi(x_j - \Delta x)) - 2\sin(k\pi x_j) + \sin(k\pi(x_j + \Delta x))\}$$

$$= -\frac{1}{\Delta x^2}\{\underbrace{\sin(k\pi x_j - k\pi\Delta x) + \sin(k\pi x_j + k\pi\Delta x)}_{2\cos(k\pi\Delta x)\sin(k\pi x_j)} - 2\sin(k\pi x_j)\}$$

The Eigenvalue Problem

Error Analysis

...Analytical Solution: $\hat{\underline{u}}^k, \hat{\lambda}^k$

Thus:

$$-\frac{1}{\Delta x^2}\{\hat{u}^k_{j-1} - 2\hat{u}^k_j + \hat{u}^k_{j+1}\}$$

$$= -\frac{1}{\Delta x^2}\{2\cos(k\pi\Delta x)\sin(k\pi x_j) - 2\sin(k\pi x_j)\}$$

$$= \underbrace{\frac{2}{\Delta x^2}\{1 - \cos(k\pi\Delta x)\}}_{\hat{\lambda}^k}\ \underbrace{\sin(k\pi x_j)}_{\hat{u}^k_j}.$$

$$A\hat{\underline{u}}^k = \hat{\lambda}^k\hat{\underline{u}}^k$$

Low modes

For fixed k, $\Delta x \to 0$:

$$\hat{\lambda}^k = \frac{2}{\Delta x^2}\{1 - \cos(k\pi\Delta x)\}$$

$$= \frac{2}{\Delta x^2}\{1 - (1 - \frac{1}{2}k^2\pi^2\Delta x^2 + \mathcal{O}(\Delta x^4))\}$$

$$= k^2\pi^2 + \mathcal{O}(\Delta x^2)$$

second-order convergence, $\hat{\lambda}^k \to \lambda^k$.

The Eigenvalue Problem

Error Analysis

...Conclusions...

High modes:

For $k = n$,

$\Delta x = \frac{1}{n+1}$

$$\hat{\lambda}^n = \frac{2}{\Delta x^2}\{1 - \cos(\frac{n\pi}{n+1})\}$$

$$= 4(n+1)^2 \quad \text{as} \quad \Delta x \to 0$$

$$\neq n^2\pi^2 = \lambda^n.$$

High modes ($k \approx n$) **are not** accurate.

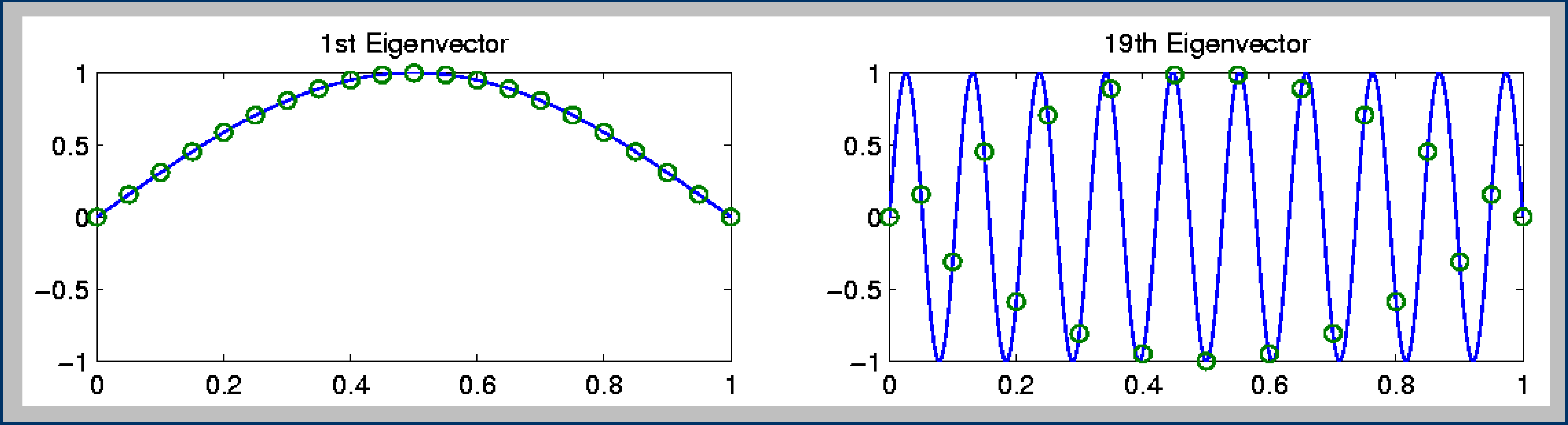
The Eigenvalue Problem

Error Analysis

...Conclusions...

Low modes vs. high modes

Example : $n = 19$, $\Delta x = 1/20$



The Eigenvalue Problem

Error Analysis

...Conclusions...

Low modes vs. high modes

$k \ll n$	$k \approx n$
$\hat{u}^k$ resolved	$\hat{u}^k$ not resolved
$\hat{\lambda}^k$ accurate	$\hat{\lambda}^k$ not accurate
$\hat{\lambda}^k - \lambda^k \sim \mathcal{O}(\Delta x^2)$	$\hat{\lambda}^k - \lambda^k$ is $\mathcal{O}(1)$

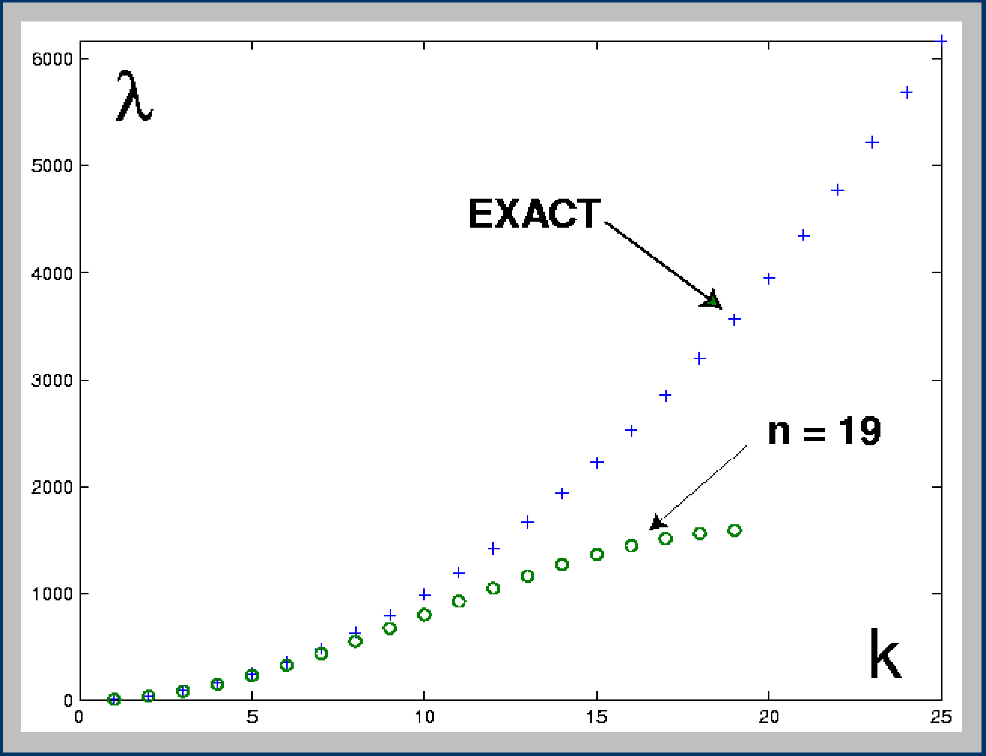
N19

BUT: as $\Delta x \to 0$, $n \to \infty$, so any fixed mode k converges.

The Eigenvalue Problem

Error Analysis

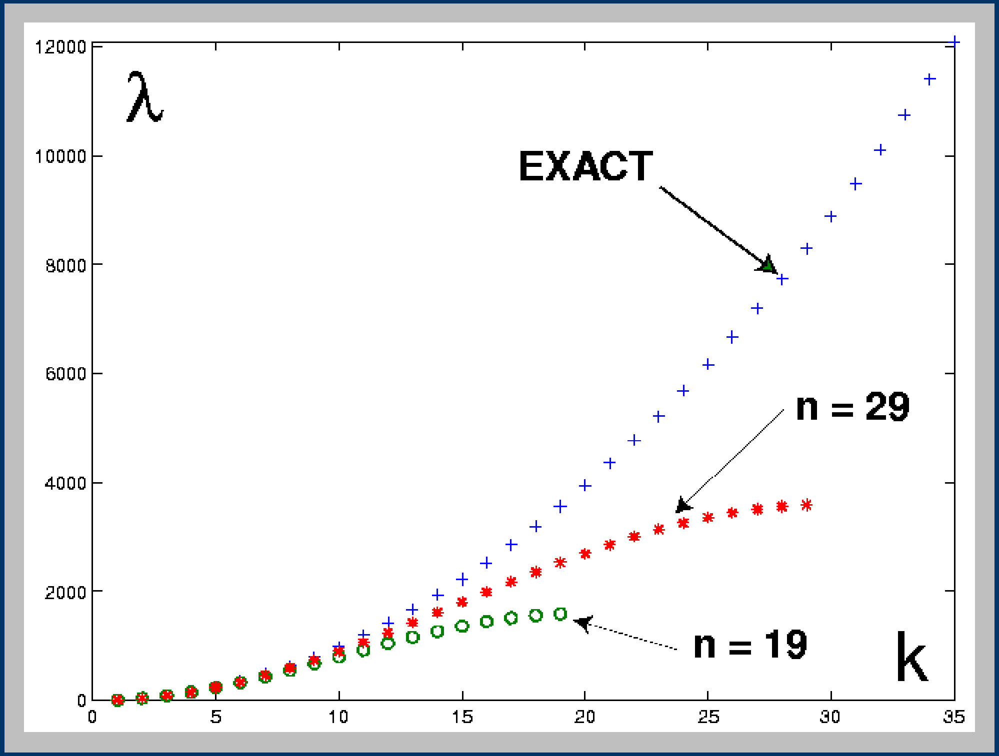
...Conclusions...



The Eigenvalue Problem

Error Analysis

...Conclusions



The Eigenvalue Problem

Condition Number of A

For a SPD matrix M, the condition number κ_M is given by

$$\kappa_M = \frac{\text{maximum eigenvalue of } M}{\text{minimum eigenvalue of } M}.$$

Thus, for our A matrix,

$$\kappa_A \rightarrow \frac{4n^2}{\pi^2} \quad \text{as} \quad \Delta x \rightarrow 0$$

grows (in $\mathbb{R}^1$) as number of grid points squared. N20

Importance: understanding solution procedures.

The Eigenvalue Problem

Link to $-u_{xx} = f$

...Discretization...

Recall: $-u_{xx} = f \quad \Rightarrow$

$$-\frac{1}{\Delta x^2}(\hat{u}_{j-1} - 2\hat{u}_j + \hat{u}_{j+1}) = f_j, \quad j = 1, \ldots, n$$

$$\hat{u}_0 = \hat{u}_{n+1} = 0$$

or

$$\boxed{A\underline{\hat{u}} = \underline{f}\ .}$$

The Eigenvalue Problem

Link to $-u_{xx} = f$

...Discretization

Error equation: $\underline{e} = \underline{u} - \underline{\hat{u}}$

$$\boxed{A\underline{e} = \underline{\tau},}$$

$$|\tau_j| \leq \max_{x \in (0,1)} \frac{\Delta x^2}{12} u^{(4)}(x) \equiv c_\tau \Delta x^2, \quad \text{for } j = 1, \ldots$$

$\rightarrow 0$ as $\Delta x \rightarrow 0$ (**consistency**).

The Eigenvalue Problem

Link to $-u_{xx} = f$

Norm Definition

We will use the "modified" $\|\cdot\|_2$ norm

N21

$$\|\underline{v}\|^2 \equiv \Delta x \sum_{i=1}^{n} \underline{v}^T \underline{v} \quad \text{for} \quad \underline{v} \in \mathbb{R}^n$$

$$\|\underline{v}\| = \sqrt{\Delta x}\|\underline{v}\|_2$$

Thus, from consistency

$$\|\underline{\tau}\| \leq c_\tau \Delta x^2.$$

The Eigenvalue Problem

Link to $-u_{xx} = f$

$\|\cdot\|$ Convergence...

Ingredients:

1. Rayleigh Quotient: **N22**

$$\hat{\lambda}^1 \leq \frac{\underline{v}^T A \underline{v}}{\underline{v}^T \underline{v}} \leq \hat{\lambda}^n, \quad \text{for all } \underline{v} \in \mathbb{R}^n$$

2. Cauchy-Schwarz Inequality: **N23**

$$\underline{v}^T \underline{w} \leq (\underline{v}^T \underline{v})^{\frac{1}{2}} \, (\underline{w}^T \underline{w})^{\frac{1}{2}} \quad \text{for all } \underline{v} \in \mathbb{R}^n$$

The Eigenvalue Problem

Link to $-u_{xx} = f$

...$\|\cdot\|$ Convergence...

Convergence proof:

$$\boxed{A\underline{e} = \underline{\tau}}$$

$$\underline{e}^T A\underline{e} = \underline{e}^T\underline{\tau}$$

$$\underbrace{\hat{\lambda}^1(\underline{e}^T\underline{e})}_{\times\Delta x} \le \underbrace{(\underline{e}^T\underline{e})^{\frac{1}{2}}}_{\Delta x^{1/2}}\underbrace{(\underline{\tau}^T\underline{\tau})^{\frac{1}{2}}}_{\Delta x^{1/2}}$$

$$\hat{\lambda}^1\|\underline{e}\|^2 \le \|\underline{e}\|\;\|\underline{\tau}\|$$

The Eigenvalue Problem

Link to $-u_{xx} = f$

...$\|\cdot\|$ Convergence...

$$\Rightarrow \|\underline{e}\| \leq \frac{1}{\hat{\lambda}^1}\|\tau\| \leq \frac{c_\tau}{\hat{\lambda}^1}\Delta x^2$$

N24 N25 N26

The Eigenvalue Problem

Link to $-u_{xx} = f$

...$\|\cdot\|$ Convergence...

Alternative Derivation

N27

Since

$$\|A^{-1}\|_2 = \frac{1}{\hat{\lambda}^1}$$

From error equation

$$\|\underline{e}\|_2 \leq \|A^{-1}\|_2 \|\underline{\tau}\|_2.$$

Multiplying by $\sqrt{\Delta x}$

$$\|\underline{e}\| \leq \frac{1}{\hat{\lambda}^1} \|\underline{\tau}\|.$$