Iterative Methods:

Multigrid Techniques

Lecture 7

Background

– Developed over the last 25 years — Brandt (1973) published first paper with practical results.

– Offers the possibility of solving a problem with work and storage proportional to the number of unknowns.

– Well developed for linear elliptic problems — application to other equations is still an active area of research.

Good Introductory Reference: *A Multigrid Tutorial*, W.L. Briggs, V.E. Henson, and S.F. McCormick, SIAM Monograph, 2000.

Basic Principles

Some ideas

1. Multigrid is an iterative method $\rightarrow$ a *good initial guess* will reduce the number of iterations:

   to solve $A_h u_h = f_h$ by iteration, we could take

   $u_h^0 \sim u_{2h}$, where $A_{2h} u_{2h} = f_{2h}$ ...

   but ... the number of iterations needed to solve $A_h u_h = f_h$ still $O(n^2)$.

$h = \frac{1}{n+1}$

Basic Principles

Some ideas

2. **If** after a few iterations, the *error is smooth*, we could solve for the error on a *coarser mesh*, e.g $A_{2h}\, e_{2h} = r_{2h}$.

   – Smooth functions can be represented on coarser grids;

   – Coarse grid solutions are cheaper.

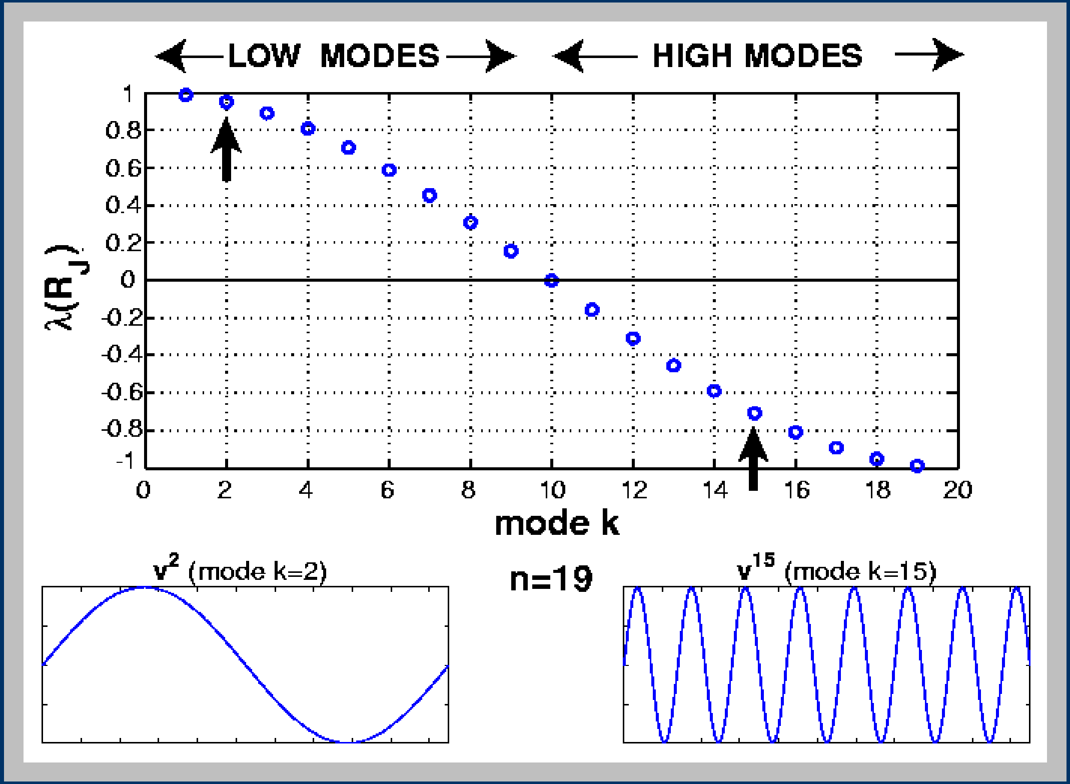
Smoother

If the *high frequency* components of the error decay faster than the *low frequency* components, we say that the iterative method is a *smoother*.

Basic Principles

Smoother

Jacobi



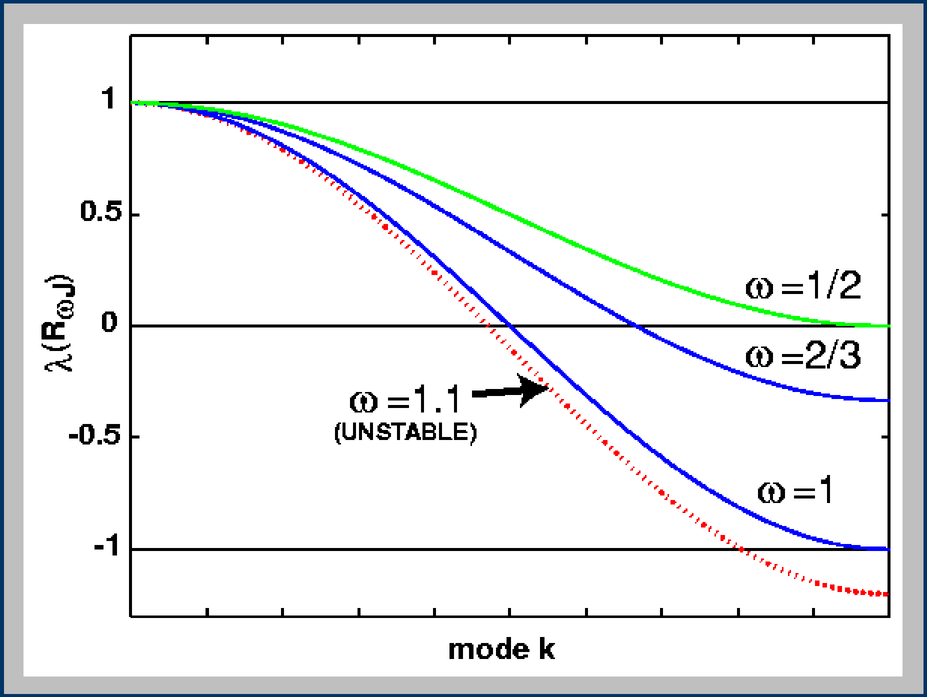
Is Jacobi a smoother? . . . $\rightarrow$ *NO*

Basic Principles

Smoother

Under-Relaxed Jacobi...

$$R_{\omega J} = \omega R_J + (1-\omega)\, I$$



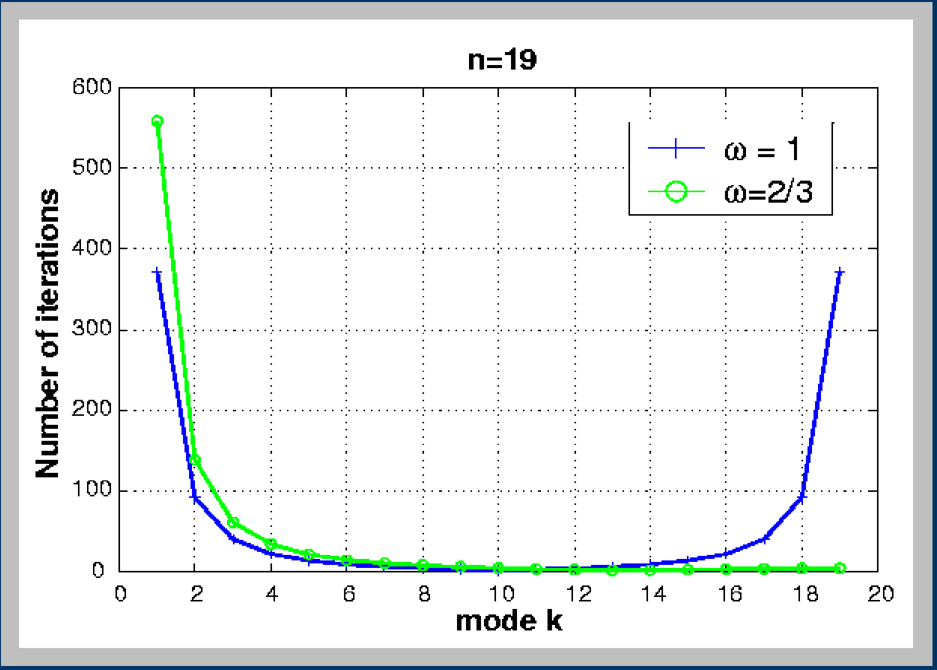
$$\lambda^k(R_{\omega J}) = \omega\lambda^k(R_J) + (1-\omega) = 1 - \omega(1-\lambda^k(R_J))\,,$$
$$k = 1, \ldots, n$$

Basic Principles

Smoother

...Under-Relaxed Jacobi

Iterations required to reduce an error mode by a factor of 100

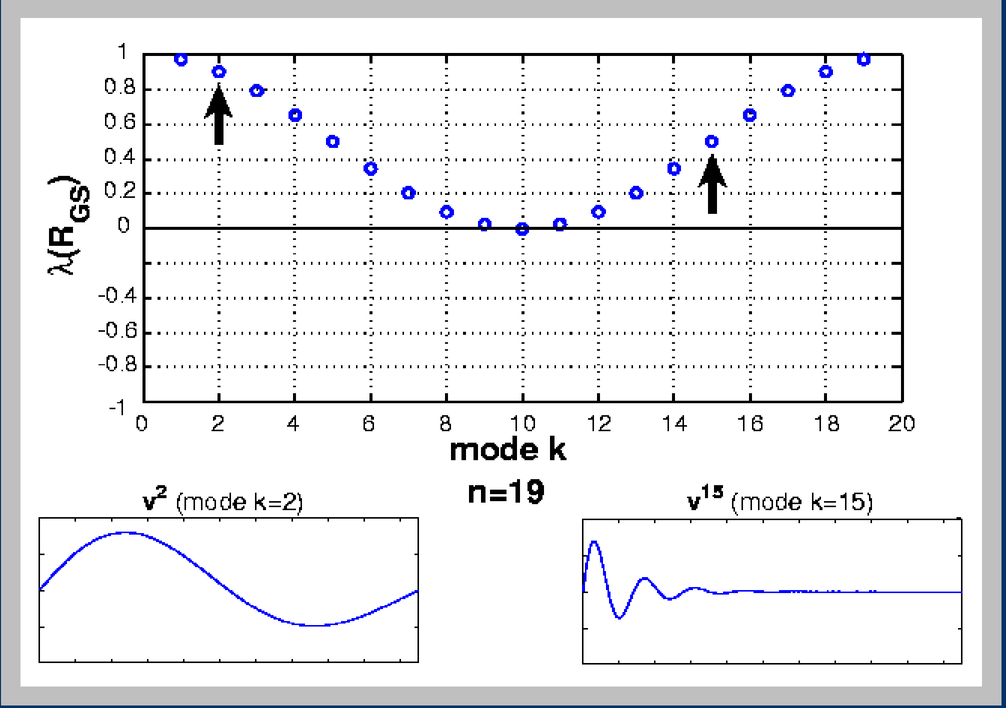


Basic Principles

Smoother

Gauss-Seidel...

Recall,



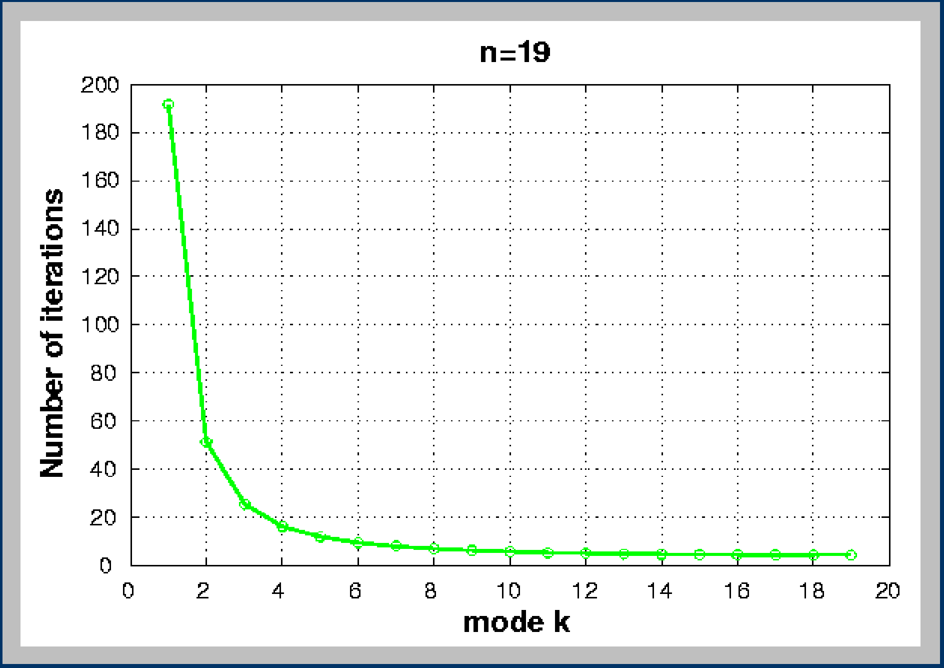
Is Gauss-Seidel a good smoother? . . .

Basic Principles

Smoother

...Gauss-Seidel

Iterations required to reduce an *A error mode* by a factor of 100



. . . **GS** is a *good smoother*.

Basic Principles

Restriction

Given w_h we obtain w_{2h} by **restriction**

$$w_{2h} = I_{2h}^{h}\, w_h$$

I_{2h}^{h}: restriction operator (matrix).

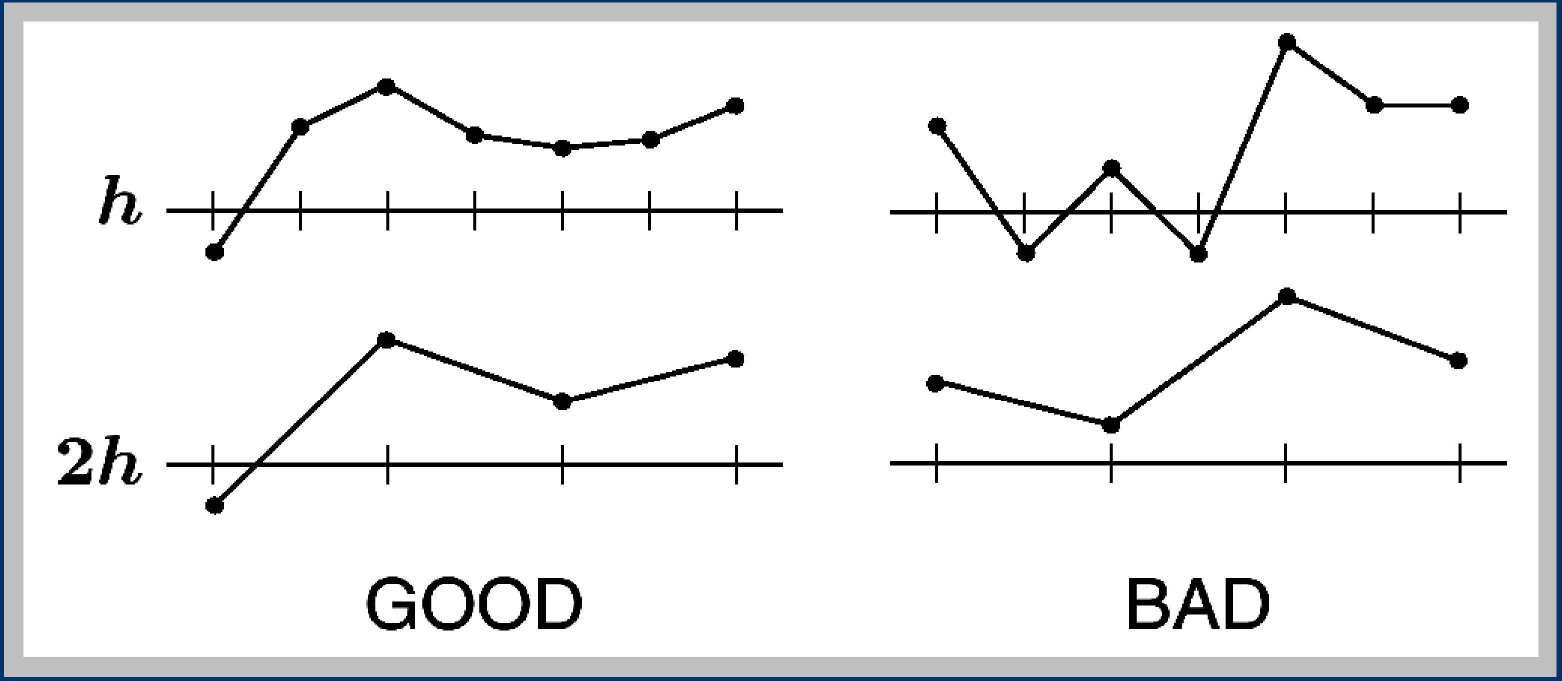
Simplest procedure is *injection*

$$w_{2h,i} = w_{h,2i} \quad \text{for } i = 1, \ldots, \frac{n-1}{2}$$

Basic Principles

Restriction

Intuitively,



Basic Principles

Restriction

If we write v^k: eigenvectors of A

$$w_h = \sum_{k=1}^{n} c_k\, v^k$$

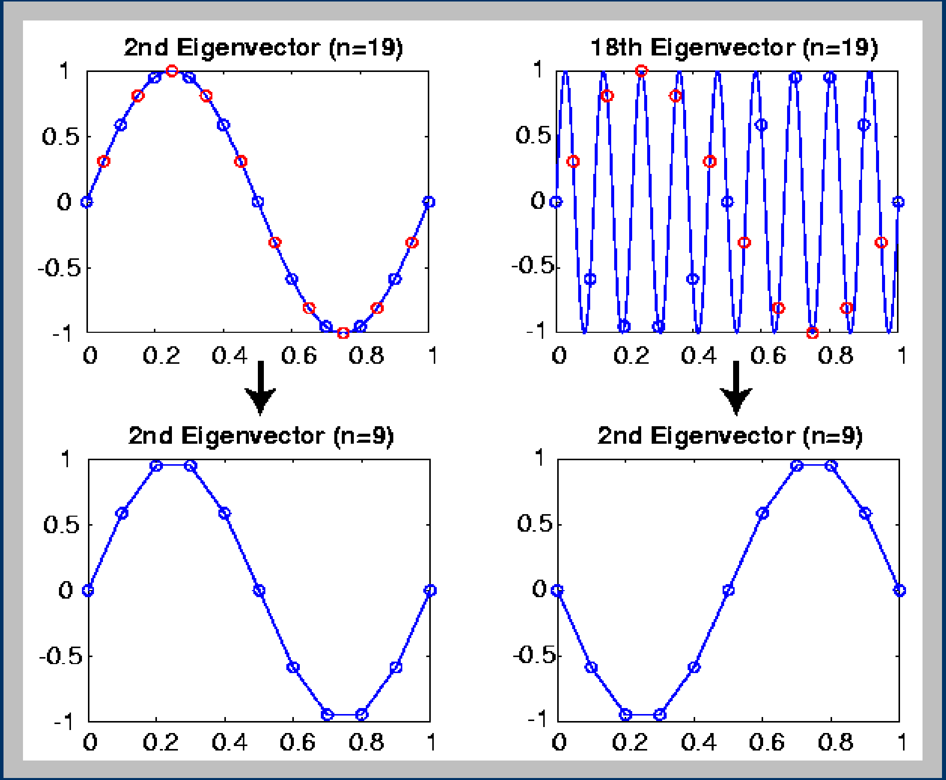
Only the modes $k = 1, \ldots, \frac{n-1}{2}$ are "visible" by grid $2h$.

$$\overbrace{1, 2, \ldots, \frac{n-1}{2}}^{\text{“visible” by grid } 2h},\ \overbrace{\frac{n+1}{2}, \ldots, n-1, n}^{\text{aliased}}$$

Basic Principles

Restriction

Aliasing



Mode $k > (n-1)/2$ on grid h becomes $(n-k)$ mode on grid $2h$.

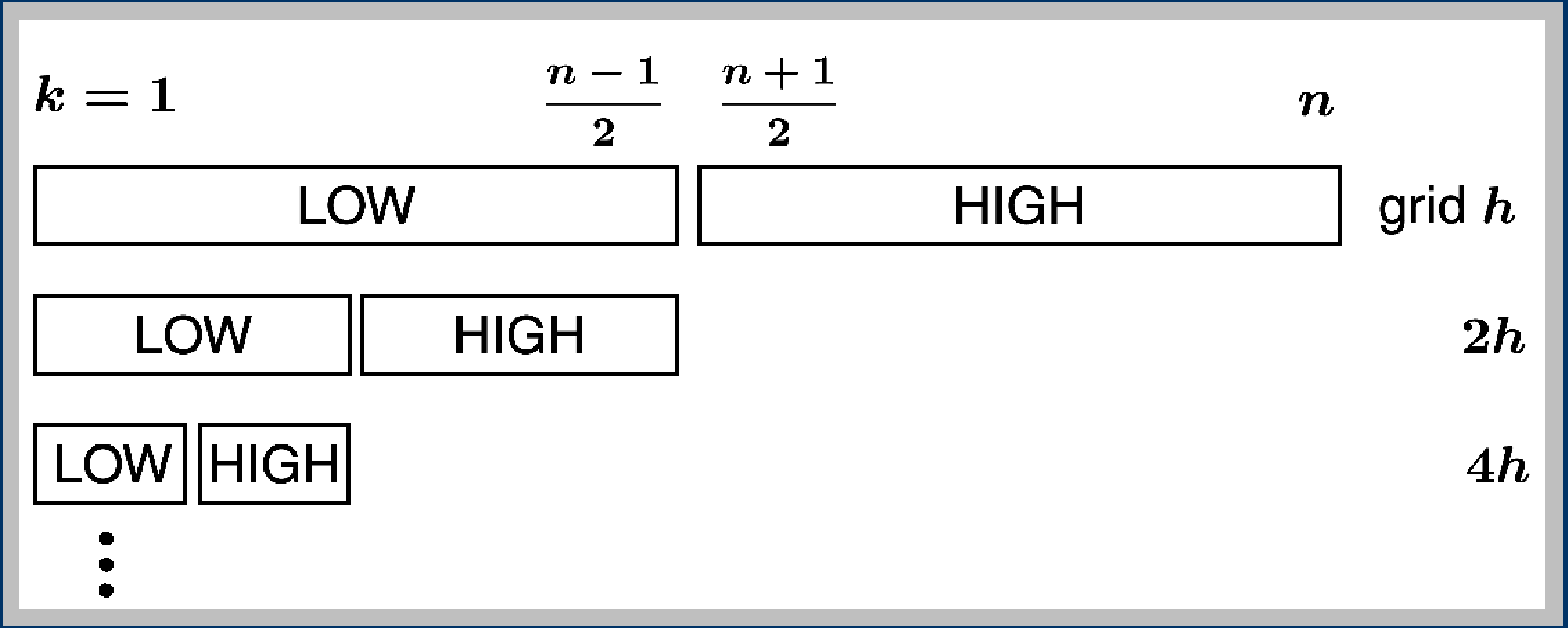
Basic Principles

Restriction

Summary

– Only low modes in h can be represented well in $2h$.

– Low modes on h become higher modes in $2h$.



Basic Principles

Prolongation

Given w_{2h} we obtain w_h by **prolongation**

$$w_h = I_h^{2h} \, w_{2h}$$

I_h^{2h}: prolongation operator (matrix).

N1

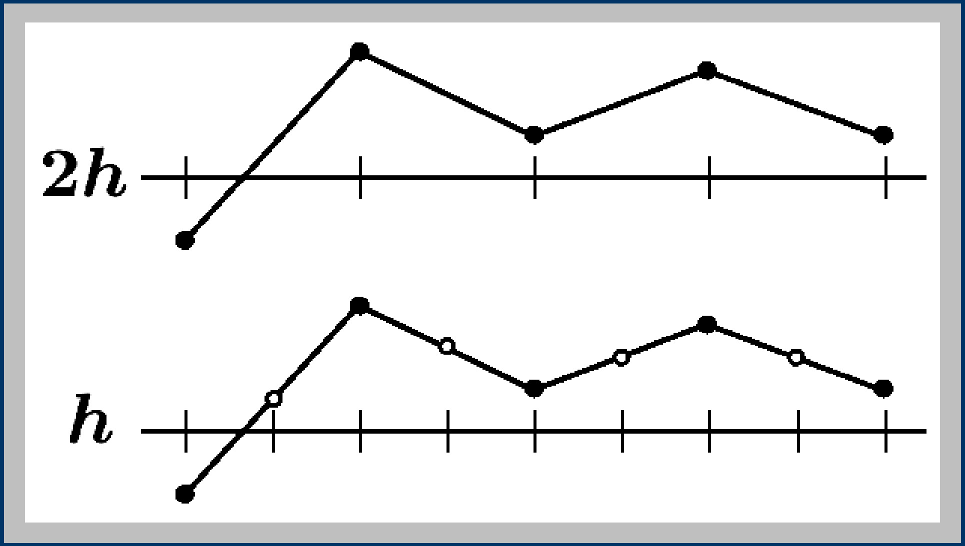
Typically, we use *interpolation*. $i = 1, \ldots, \frac{n-1}{2}$

$$w_{h,2i} = w_{2h,i}$$

$$w_{h,2i+1} = \frac{1}{2}\left(w_{2h,i} + w_{2h,i+1}\right)$$

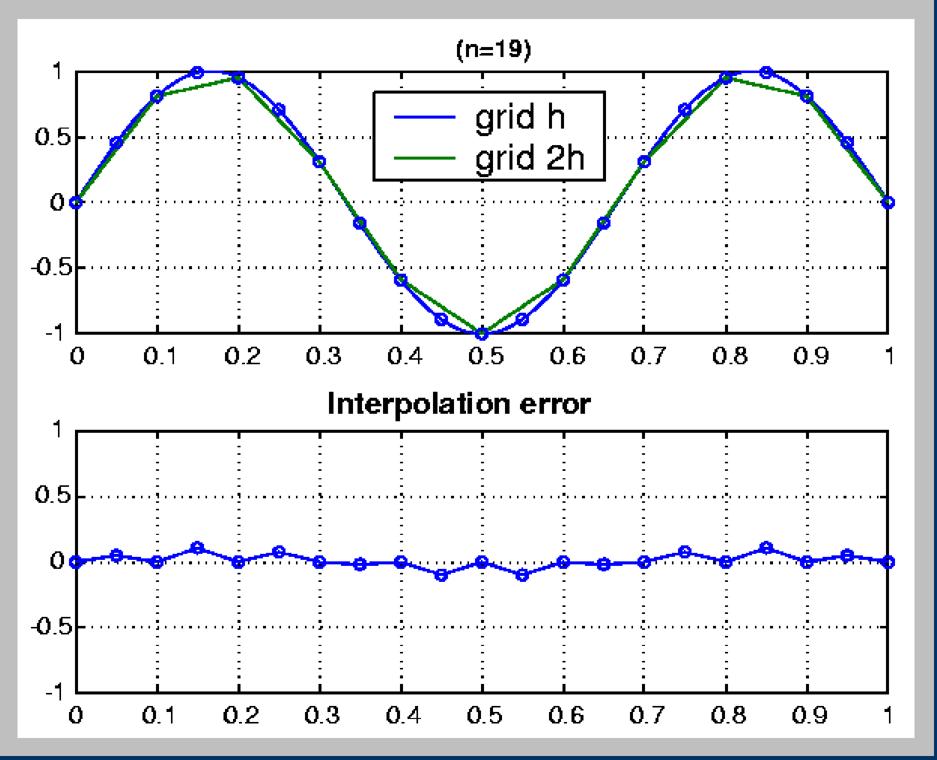
Basic Principles

Prolongation



Basic Principles

Interpolation Error



Interpolation introduces high frequency errors.

Two Grid (Correction) Scheme

One cycle $u_h^{r+1} \leftarrow MG(u_h^r, f_h)$

– *Relax* ν_1 iterations of $A_h u_h = f_h$ with initial guess $u_h^r \rightarrow u_h^{r+1/3}$.

– Compute $r_h = f - A_h u_h^{r+1/3}$, and *restrict* $r_{2h} = I_{2h}^h r_h$.

– Solve $A_{2h} e_{2h} = r_{2h}$ on $2h$.

– *Prolongate* $e_h = I_h^{2h} e_{2h}$, and correct $u_h^{r+2/3} = u_h^{r+1/3} + e_h$.

– *Relax* ν_2 iterations of $A_h u_h = f_h$ with initial guess $u_h^{r+2/3} \rightarrow u_h^{r+1}$.

Two Grid (Correction) Scheme

Example

We solve $u(0) = u(1) = 0$

$$-u_{xx} = -25(\sin(5\pi x) + 9\sin(15\pi x)) \; .$$

Initial guess: $u^0 = 0$

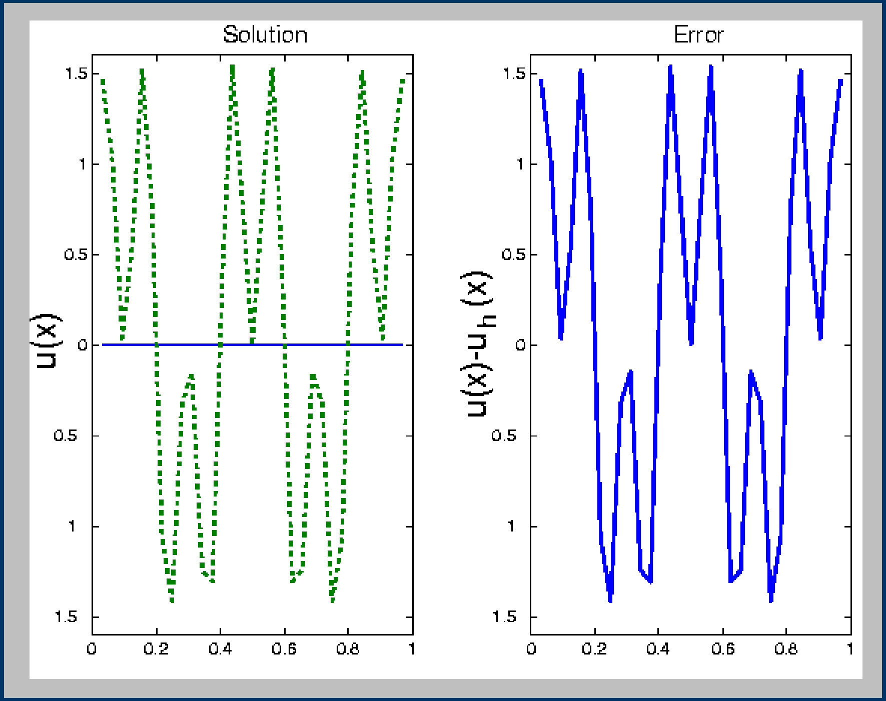
Solution: $u = \sin(5\pi x) + \sin(15\pi x)$

Two grid scheme: $h = \frac{1}{32}$, $2h = \frac{1}{16}$

Solve using under-relaxed Jacobi with $\omega = \frac{2}{3}$

Two Grid (Correction) Scheme

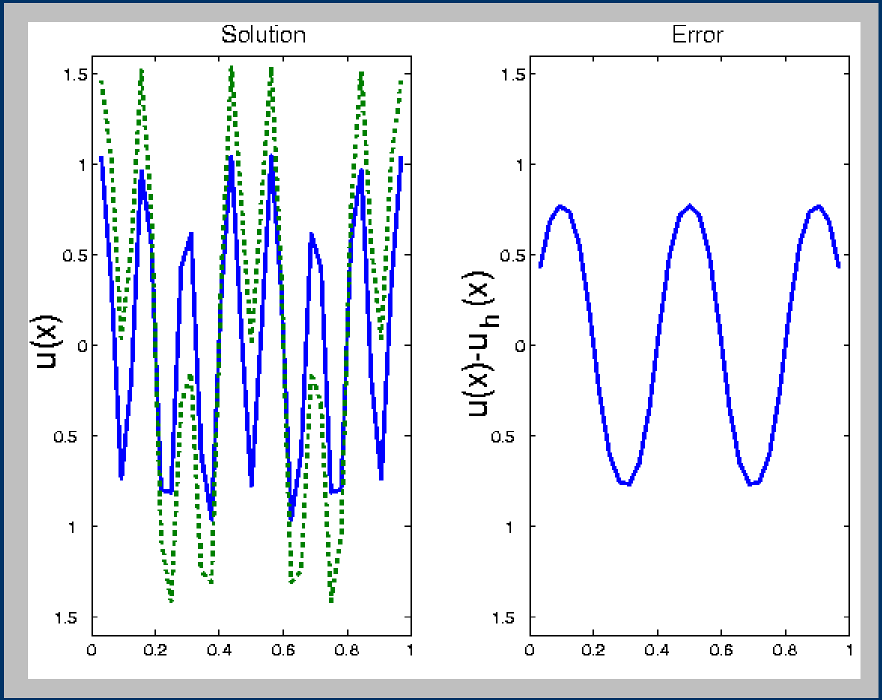
Initial condition



Two Grid (Correction) Scheme

Example

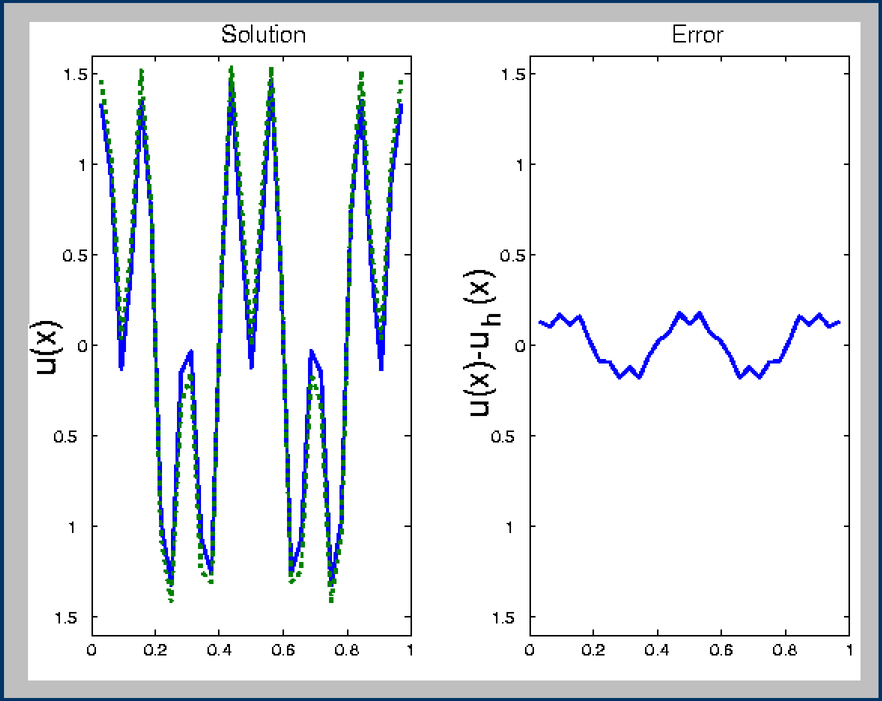
After $\nu_1 = 2$ iterations on the fine mesh



Two Grid (Correction) Scheme

Example

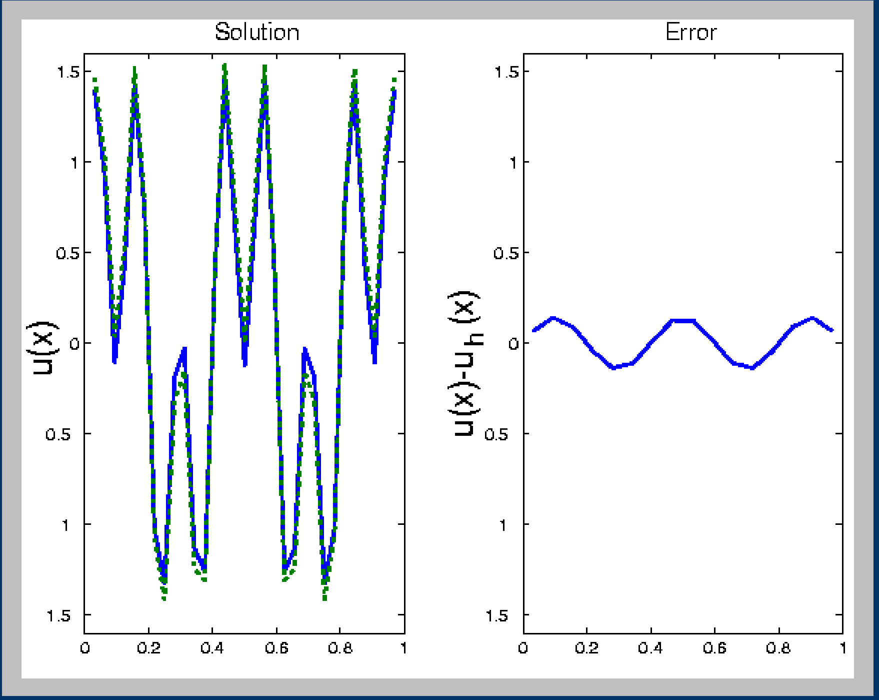
After coarse grid correction (4 iterations)



Two Grid (Correction) Scheme

Example

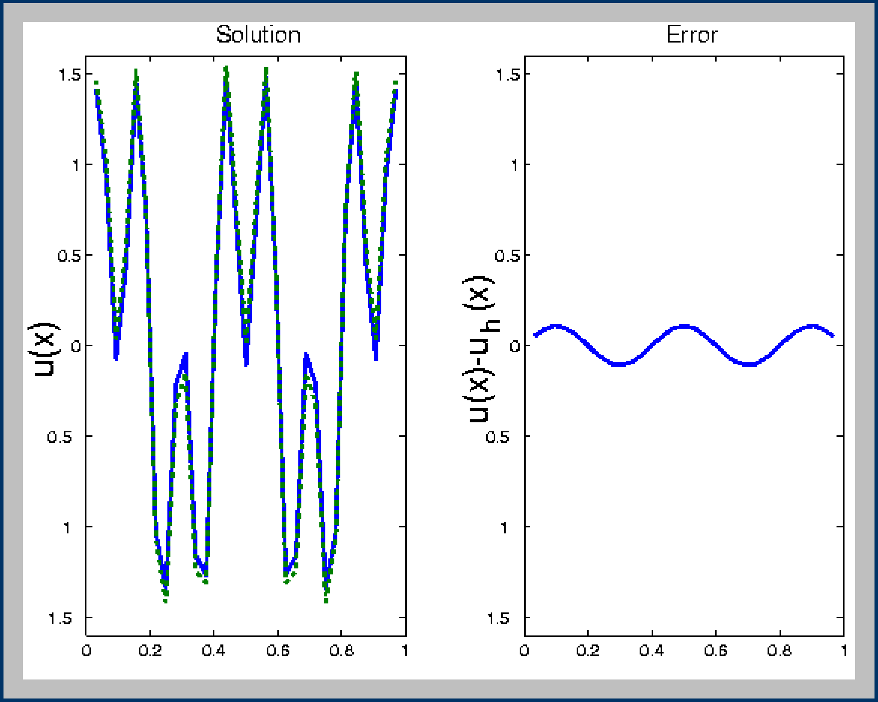
After $\nu_2 = 2$ post smoothing iterations (end of cycle 1)



Two Grid (Correction) Scheme

Example

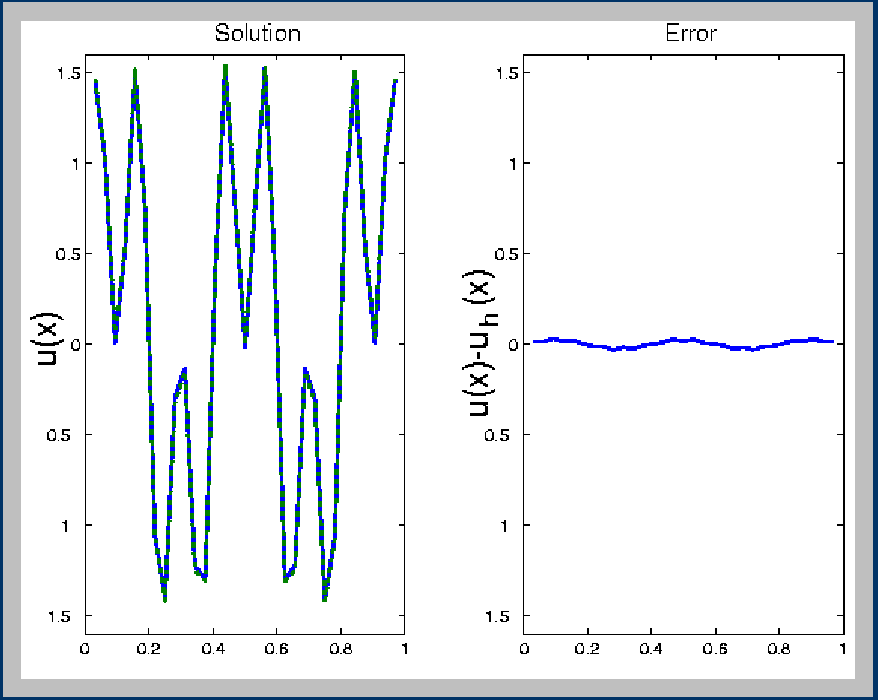
After $\nu_1 = 2$ iterations



Two Grid (Correction) Scheme

Example

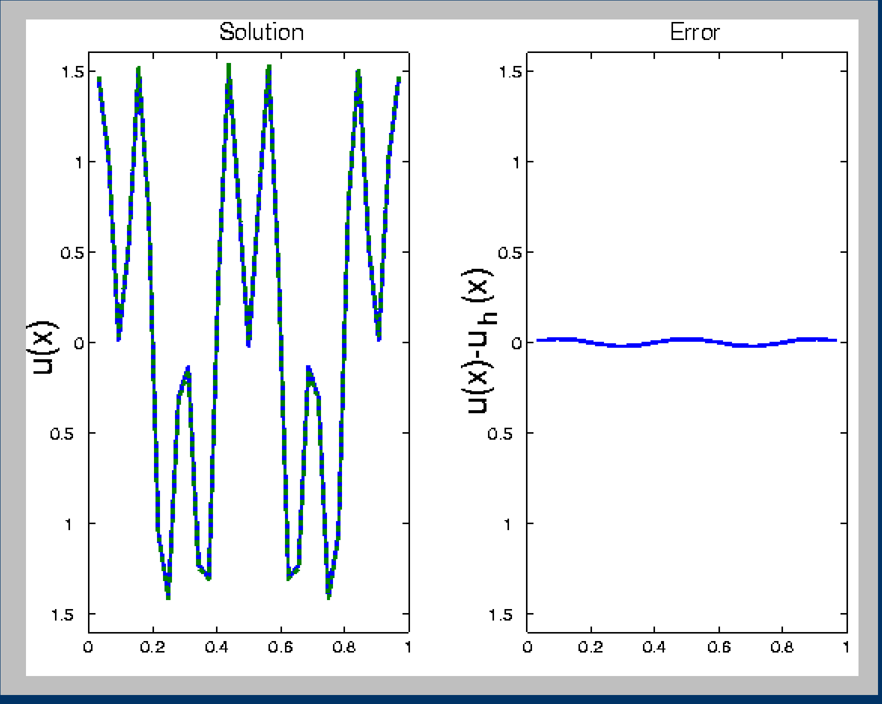
After coarse grid correction



Two Grid (Correction) Scheme

Example

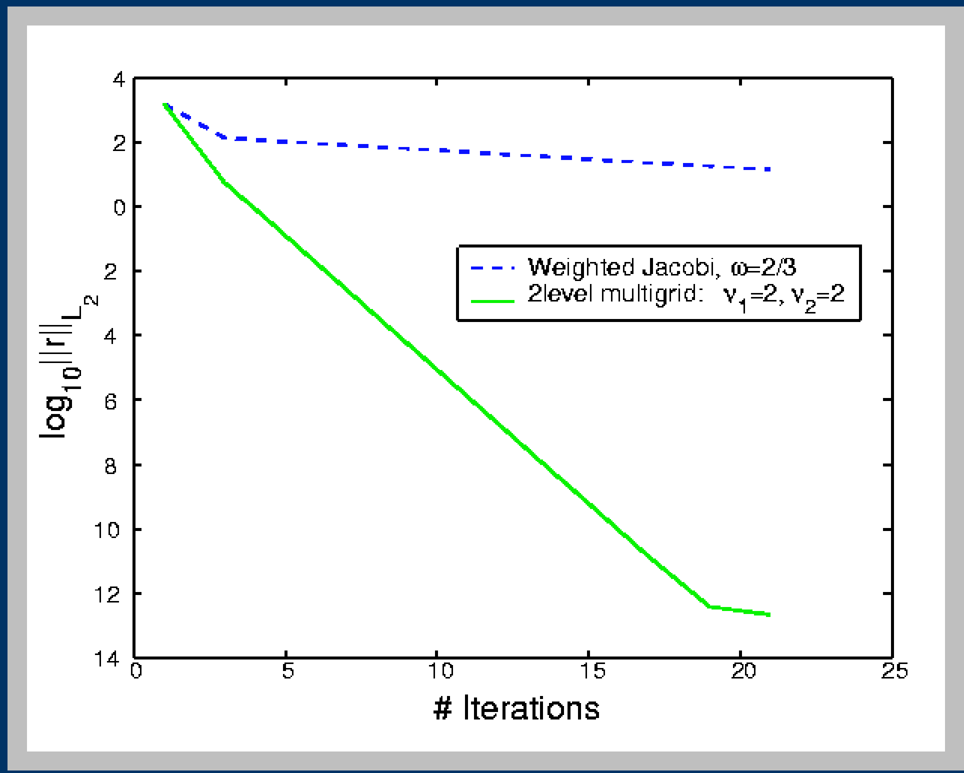
After $\nu_2 = 2$ iterations (end of cycle 2)



Two Grid (Correction) Scheme

Example

Mutligrid convergence vs. single grid



Multiple Grids

V-Cycle

One cycle $u_h^{r+1} \leftarrow VG_h(u_h^r, f_h)$

– Relax ν_1 times on $A_h\, u_h = f_h$ with initial guess $u_h^r \rightarrow u_h^{r+1/3}$.

– If $h \equiv$ coarsest grid, go to (SKIP)

Else $r_{2h} \leftarrow I_{2h}^h(f_h - A_h\, u^{r+1/3})$

$e_{2h} \leftarrow VG_{2h}(0, r_{2h})$.

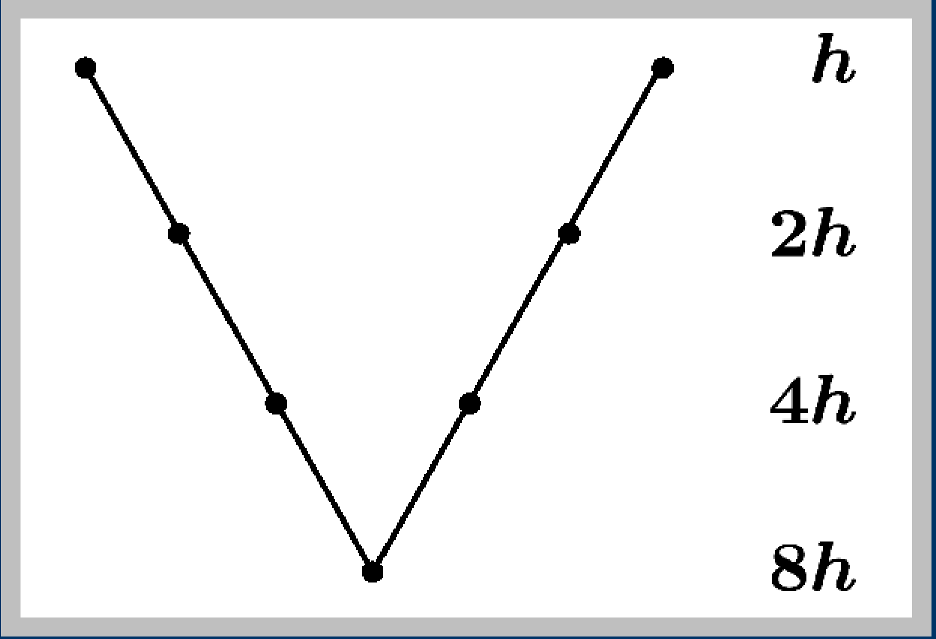
– Correct $u_h^{r+2/3} = u_h^{r+1/3} + I_h^{2h}\, e_{2h}$.

– (SKIP) Relax ν_2 times on $A_h\, u_h = f_h$ with initial guess $u_h^{r+2/3} \rightarrow u_h^{r+1}$.

Multiple Grids

V-Cycle

Schematically



Multiple Grids

V-Cycle

2D Example...

Solve

$$-(u_{xx} + u_{yy}) = 1, \qquad \in \Omega \equiv \text{unit square}$$

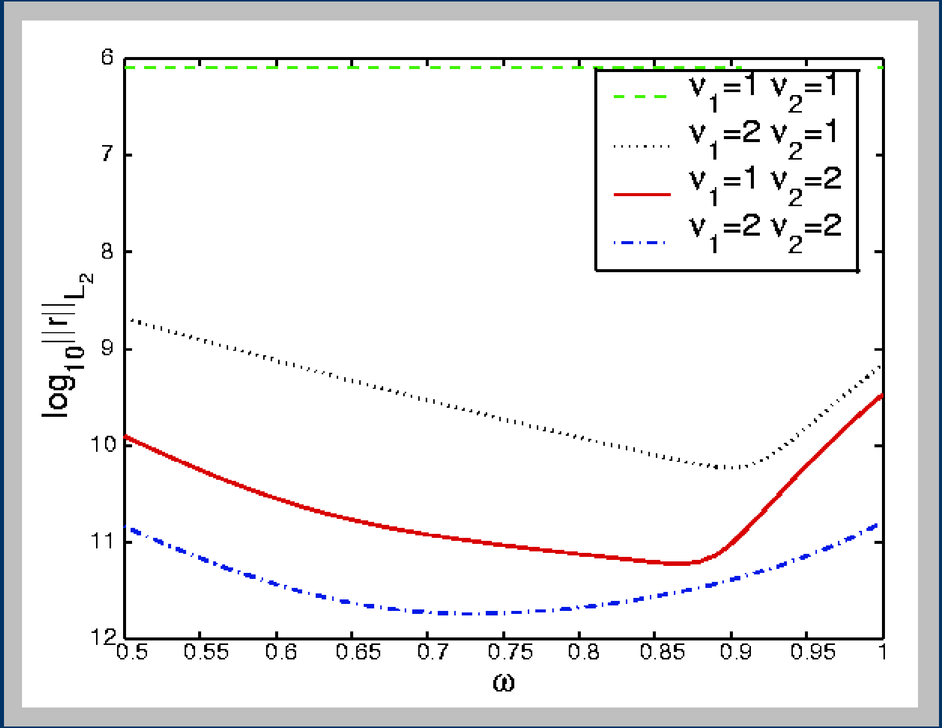
$$u = 0 \qquad \text{on the boundary}$$

Multiple Grids

V-Cycle

...2D Example...

Parameter dependence

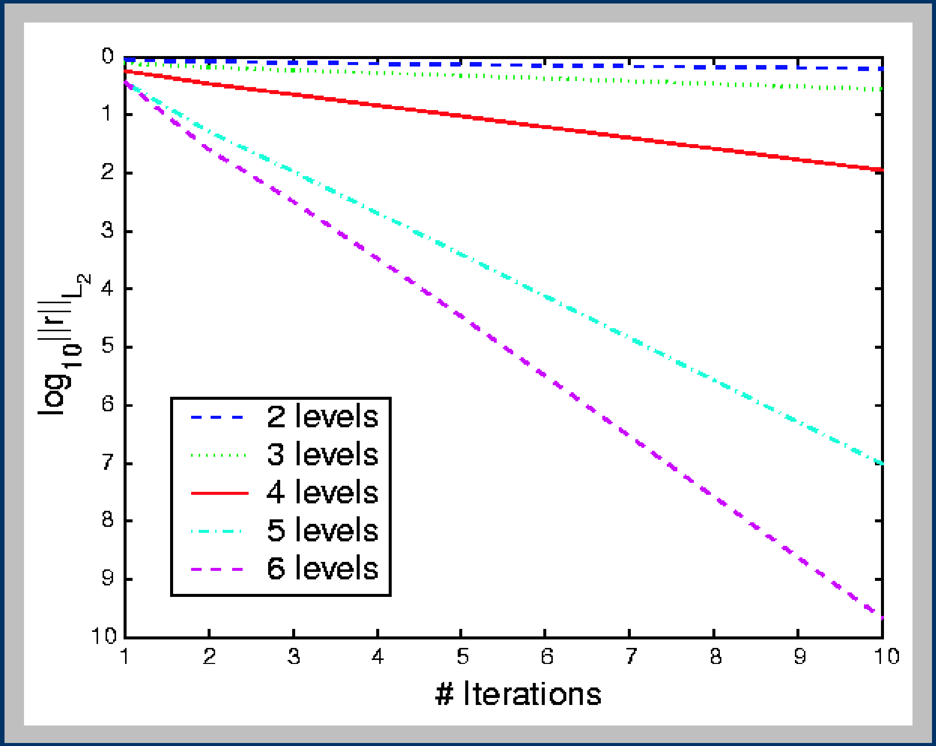


Multiple Grids

V-Cycle

...2D Example...

Convergence as a function of grid levels (same fine mesh)

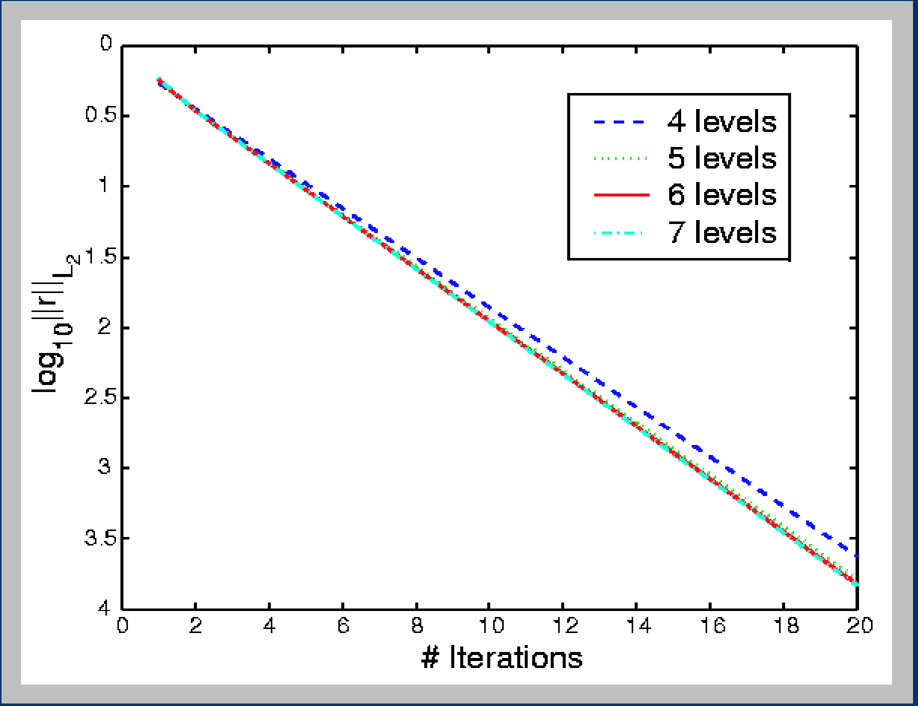


Multiple Grids

V-Cycle

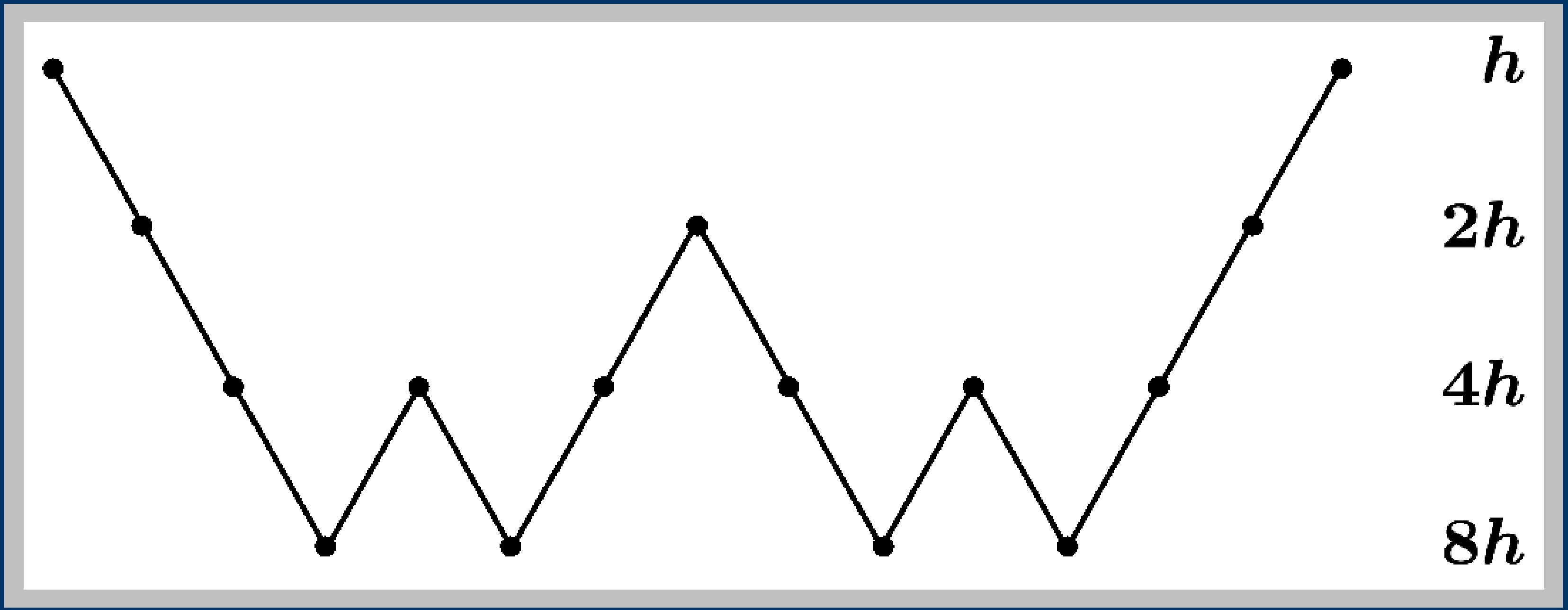
...2D Example

Convergence as a function of grid levels (same coarse mesh)



Multiple Grids

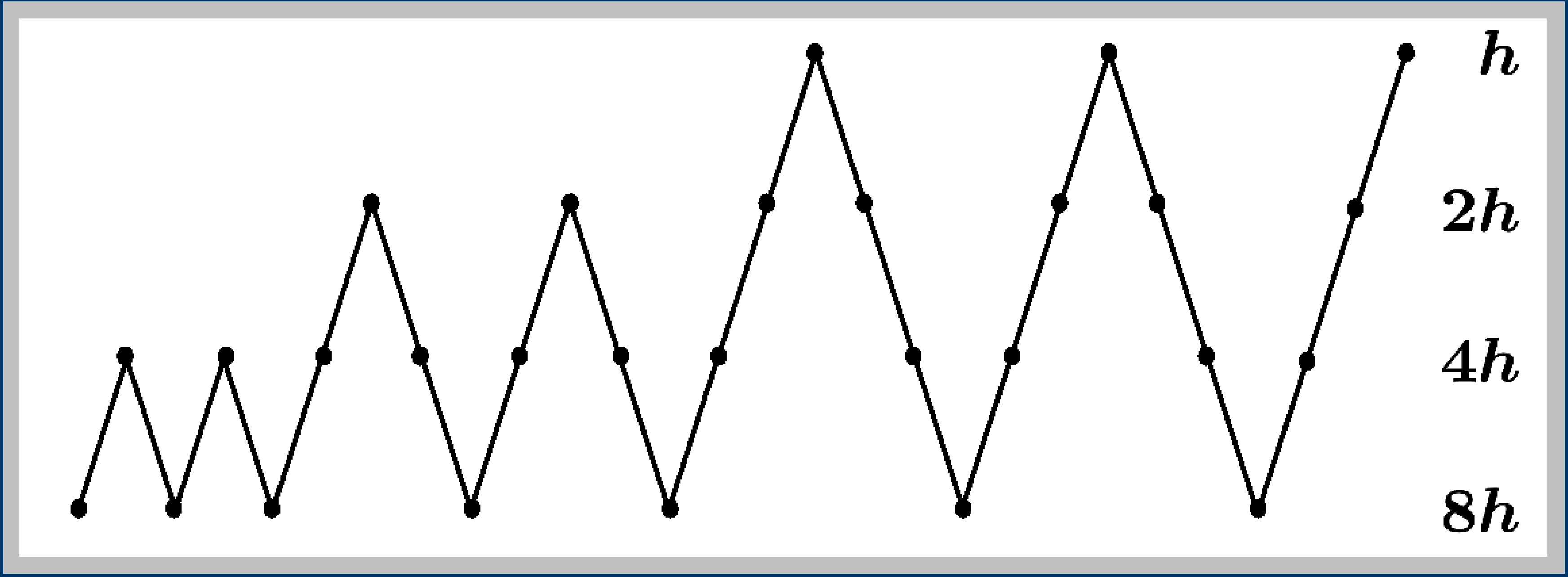
W-Cycles



Full Multigrid Scheme

Schematically

Putting it all together . . .



More Advanced Topics

- Anisotropic grids/equations.
- Algebraic multigrid.
- Convergence theory.
- How to deal with other operators.