Last time: poroelasticity
continuum theory accounting for fluid - structure interactions
Solid: Hookean (linear), incompressible, uniform, isotropic, inertia - free
Fluid: incompressible, inviscid (but solid- fluid friction), Newtonian, inertia - free ($Re \simeq 0$)

- Governing equations

• constitutive law $\quad \sigma_{11} = H\,\varepsilon_{11} - p \quad$ 1D, confined compression experiment

$$H = 2G + \lambda$$

• fluid-structure interactions $\quad U_1 = -k\,\dfrac{\partial p}{\partial x_1} \quad (2a)$

k: hydraulic permeability

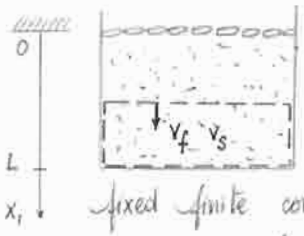
• conservation of mass = - from volume flux $\quad \bar{U} = \phi\,(\bar{v}_f - \bar{v}_s)$

$$\bar{v}_s = \frac{\partial \bar{u}}{\partial t}$$

$\bar{v}_f$ and $\bar{v}_s$ are averaged over a mesoscopic length scale L

$L \gg$ pore size, molecule
$L \ll$ macroscopic tissue

* from 1.D compaction

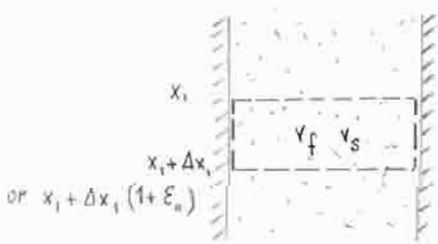


fixed finite control volume

$$\left.\begin{aligned} A_f\, v_s + A_s\, v_s &= 0 \\ U_1 &= \phi\,(v_f - v_s) \\ v_s &= \frac{\partial u_1}{\partial t} \end{aligned}\right\} \quad U_1 = -\frac{\partial u_1}{\partial t}$$

* even in steady state, without dynamic compression, there should be a relationship between U_1 and u_1

net flow into control volume = change in volume of control volume



deformable control volume
always contains the same solid elements

$$\Big[\underbrace{A_f\,(v_f - v_s)}_{\downarrow\ \frac{\partial u_1}{\partial t}}\Big]_{x_1+\Delta x_1}^{x_1} \Delta t = \Delta x_1 \Big[A_{tot}\Big(1 + \underbrace{\frac{\partial u_1}{\partial x_1}}_{\varepsilon_{11}}\Big)\Big]_{t_0}^{t_0+\Delta t}$$

and divide by $\Delta t \cdot \Delta x_1 \cdot A_{TOT}$

$$-\frac{\partial U_1}{\partial x_1} = \frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t}\right) = \frac{\partial \varepsilon_{11}}{\partial t} = \frac{\partial v_s}{\partial x_1}$$

$$\boxed{\begin{aligned} U_1 &= -\frac{\partial u_1}{\partial t} + U_0 & (3a) \\ \bar{\nabla}\cdot\bar{U} &= -\bar{\nabla}\cdot\bar{v}_s = \bar{\nabla}\cdot\left[\phi\,(\bar{v}_f - \bar{v}_s)\right] & (3) \end{aligned}}$$

U_0 mean, steady flow

• conservation of momentum (from before) neglecting inertia

$$\frac{\Sigma F_i}{V} = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{m}{V}\,a_i = 0 \quad \text{or} \quad \boxed{\begin{aligned} \bar{\nabla}\cdot\bar{\bar{\sigma}} &= 0 & (4) \\ \frac{\partial \sigma_{11}}{\partial x_1} &= 0 & (4a) \end{aligned}}$$

Our 1D equations are

$$
\boxed{
\begin{aligned}
&(1a) \quad \sigma_{11} = H\,\varepsilon_{11} - p \\
&\qquad\ \ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \\
&(2a) \quad U_1 = -k\,\frac{\partial p}{\partial x_1} \\
&(3a) \quad U_1 = -\frac{\partial u_1}{\partial t} + U_0 \\
&(4a) \quad \frac{\partial \sigma_{11}}{\partial x_1} = 0
\end{aligned}
}
$$

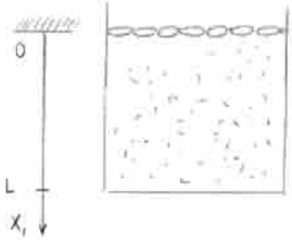
starting at

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0 = H\,\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial p}{\partial x_1} = H\,\frac{\partial^2 u_1}{\partial x_1^2} - \frac{1}{k}\cdot\frac{\partial u_1}{\partial t}$$

$$\boxed{\frac{\partial u_1}{\partial t} = Hk\,\frac{\partial^2 u_1}{\partial x_1^2} \qquad \text{"}\mathcal{D}\text{"} = Hk \ \text{diffusivity}}$$

diffusion equation

⊛ Example of stress relaxation



step displacement u_0, find $u_1(x_1,t)$, $\varepsilon_{11}(x_1,t)$, $p(x_1,t)$

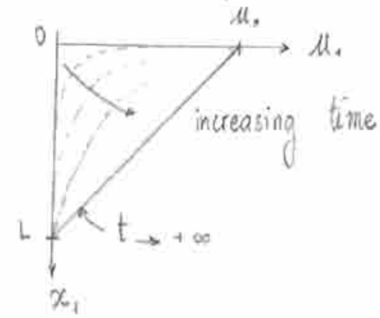
boundary conditions $u_1(x_1=0,\ t>0) = u_0$
$u_1(x_1=L,\ t>0) = 0$
initial condition $u_1(x_1,\ t=0) = 0$

Use governing equation $\frac{\partial u_1}{\partial t} = Hk\,\frac{\partial^2 u_1}{\partial x_1^2}$ to find solution (Fourier series of the form)

- analogies for relaxation time τ

poroelasticity	u_1	$\frac{L^2}{Hk}$
mass conservation	c	$\frac{L^2}{\mathcal{D}}$
momentum	v_1	$\frac{L^2}{\nu}$

$\nu \longrightarrow$ kinematic viscosity in fluid mechanics



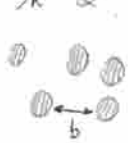
The solution will be:

$$u_1(x_1,t) = u_0\left(1 - \frac{x_1}{L}\right) - \sum_n A_n \sin\left(\frac{n\pi x_1}{L}\right)\exp\left(-\frac{t}{\tau_n}\right)$$

$$\tau_n = \frac{L^2}{n^2\pi^2\,Hk}$$ and approach to steady state with $\tau_1 = \frac{L^2}{\pi^2\,Hk}$

For cartilage, typical values are
$L \simeq 1$ mm
$H = 1$ MPa
$k = 10^{-15}\ m^4\,N^{-1}s^{-1}$
} $\tau_{n=1} \simeq 100$ s

note: if $k = \frac{k'}{\mu}$, $k' \propto b^2$

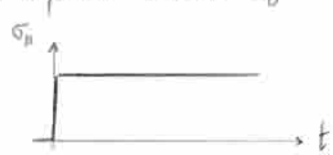


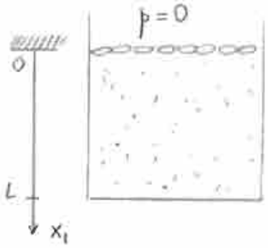
then $k' = k\mu \simeq 10^{-15}\,10^{-3} \simeq 10^{-18}\ m^2 \Longrightarrow b \simeq 1$ nm
μ viscosity of water

for a sponge, $b \simeq 0.1$ mm

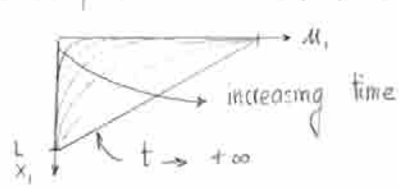
⊛ Example of creep

Given imposed load σ_0





find displacement $u_1(x_1,t)$, $\varepsilon_{11}(x_1,t)$



Boundary conditions $u_1 (x_1 = L, t) = 0$

$\frac{\partial u_1}{\partial x_1} (x_1 = 0, t) = \frac{\sigma_0}{H} = \text{constant}$ $\equiv$ constant flux $\equiv$ constant shear (Couette)

⊛ Example of dynamic compression in 1D

$$\frac{\partial u_1}{\partial t} = Hk \frac{\partial^2 u_1}{\partial x_1^2} \quad (E)$$

excitation $u_1 (x_1 = 0, t) = u_0 \cos(\omega t)$

look for solution $u_1 (x_1, t) = \mathcal{Re} [\hat{u}_1 \exp(i\omega t)]$, $\hat{u}_1$ complex

(E) becomes $i\omega \hat{u}_1 = Hk \frac{d^2 \hat{u}_1}{d x_1^2}$

BC $\hat{u}_1 (x_1 = 0) = u_0$

$\hat{u}_1 (x_1 = L) = 0$

Get solution of the form

amplitude envelop.

$$u_0 \frac{\sinh\left(\sqrt{\frac{i\omega}{Hk}} (L - x_1)\right)}{\sinh\left(\sqrt{\frac{i\omega}{Hk}}\right)}$$

there are some frequencies above which the oscillations are not penetrating all the way to the bottom

depth penetration $\delta \sim \sqrt{Hk\,\tau} \sim \sqrt{\frac{Hk}{\omega}}$

(analogy for mass transport $\delta \sim \sqrt{Dt}$)

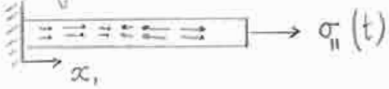
▷ for a poro-viscoelastic model

Hk becomes $\left(\frac{H_1 + i\omega\beta}{1 + i\omega\alpha}\right) k$ as before we had $G \rightarrow \frac{k_1 + i\omega\beta}{1 + i\omega\alpha}$ for the SLS

◇ Notes on simplifying assumptions

- neglecting inertia for the solid $\frac{\Sigma \bar{F}}{V} = \bar{\nabla} \cdot \bar{\bar{\sigma}} = \frac{m}{V} \bar{a} = \rho \frac{\partial^2 \bar{u}}{\partial t^2}$

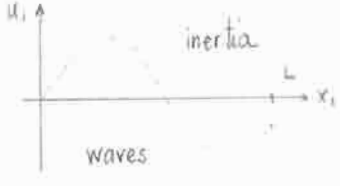
1D linearly elastic material



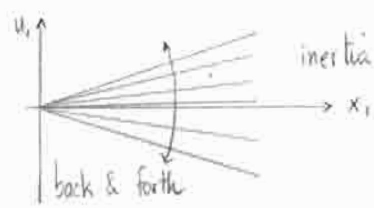
$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} = E \frac{\partial^2 u_1}{\partial x_1^2}$$

wave equation , $v_\phi = \sqrt{E/\rho}$

$\begin{cases} E \text{ Young's modulus} \\ \sigma_{11} = E\,\varepsilon_{11} = E \frac{\partial u_1}{\partial x_1} \end{cases}$



or quasi steady (slow enough) ?



if $\frac{L}{v_p} \ll \tau = \frac{1}{\omega}$, $\frac{L\omega}{v_p} \ll 1$, $\frac{L}{\lambda} \ll 1$

↳ wavelength

neglect inertia if $\frac{L}{v_p} \ll 3 \cdot 10^{-4}$ s for $v_p = \sqrt{\frac{10^6}{10^3}}$ 30 $m \cdot s^{-1}$