9.641 Neural Networks
Problem Set 8: PCA, k-means and Ring network

(Due before class on Thursday, Apr. 21)

1. Double-ring network model of a neural integrator.

   Consider a network with the following dynamics:

$$\dot{\mathbf{x}}_r + \mathbf{x}_r = \left[\mathbf{W}^r \mathbf{x}_r + \mathbf{W}^l \mathbf{x}_l + b_r\right]^+$$
$$\dot{\mathbf{x}}_l + \mathbf{x}_l = \left[\mathbf{W}^r \mathbf{x}_r + \mathbf{W}^l \mathbf{x}_l + b_l\right]^+$$

   where $\mathbf{x}_r$ and $\mathbf{x}_l$ are vectors representing the neural activities of the right and left rings respectively. The synaptic interaction matrices $\mathbf{W}^r$ and $\mathbf{W}^l$ are defined as:

$$W^r_{ij} = J_0 + J_2 \cos(2\pi(i-j)/N - \phi)$$
$$W^l_{ij} = J_0 + J_2 \cos(2\pi(i-j)/N + \phi)$$

   Simulate this network in MATLAB using $N = 100$ $J_0 = -9.8/N$, $J_2 = 10/N$, and $\phi = \pi/6$. Plot the activity fo the network using two bars graphs, one for $\mathbf{x}_r$ and another for $\mathbf{x}_l$. Show the simulation results for the following two cases:

   (a) Take $b_r = b_l = 1$.

   (b) Take $b_r = 1$ and vary b_l. You should see a travelling pulse whenever $b_l \neq b_r$. Plot the relationship between the velocity of the travelling pulse and the input b_l.

2. Principal component analysis using the CBCL face dataset.

   Download the subset of the CBCL face dataset (`faces.mat`). The faces are the columns of the matrix. (You can try to visualize faces by reshaping the columns of the matrix to be 19x19 with the `reshape` command.)

   (a) Use the `cov` command to compute the covariance of all the faces in `faces.mat`. Then use the `eig` command to find the eigenvectors and eigenvalues. Sort the eigenvectors by decreasing eigenvalues and display all eigenvectors as images (you can use the command `montage`). Describe what the top eigenvectors are coding for. What can you say about the eigenvectors as the eigenvalues decrease?

   (b) Using the `eig` command is inefficient because it finds all the eigenvectors instead of just the ones that you want. Show that iterating

$$x := \frac{Cx}{|Cx|}$$

   converges to the principal eigenvector from almost all initial conditions. Simulate it numerically and verify that the result is the same as the principal eigenvector obtained using `eig`.

3. Clustering with K-means.

   Download the matlab file called `kmeans.mat`. The data to cluster is contained in the vector `X` and the true data label in the vector `label`.

   (a) Implement the kmeans algorithm (do not use the built-in matlab function) and run it on the data with k = 3. How similar is the result of your algorithm to the true labels? What happens if you set k = 2, 4, 5? Submit your code and relevant plots.

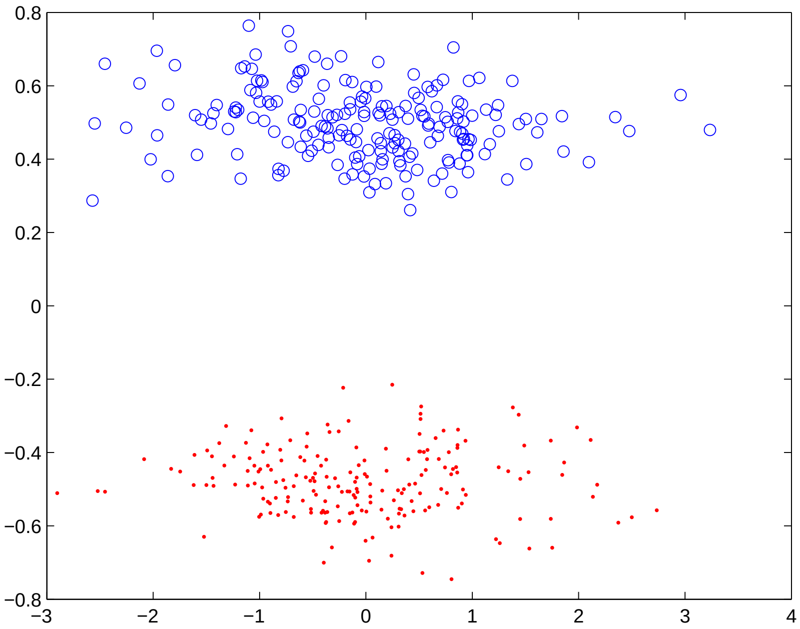

Figure 1: Data to be clustered

(b) Consider Fig. 1. What would happen if you were to run kmeans on this dataset? Explain your answer.

(c) Give examples of datasets where kmeans has multiple local minima.
Hint: Consider a toy example of three points.

4. Some statistics

(a) For two random variables x and y, prove that the magnitude of the correlation is upper bounded by the square root of the product of the second moments,

$$|\langle xy\rangle| \leq \sqrt{\langle x2\rangle}\sqrt{\langle y2\rangle}$$

Hint: Use the Cauchy-Schwarz inequality.

(b) Pearson's famous correlation coefficient r is defined as the ratio of the covariance to the product of the standard deviations. Show that the above inequality implies that

$$-1 \leq r \leq 1$$

Under what conditions is $r = 1$ or $r = -1$ exactly? The correlation coefficient assesses the quality of a least squares fit of a linear model relating x and y.