5.74, Problem Set #1
Spring 2004
Due Date: February 18, 2003

1. Let the eigenfunctions and eigenvalues of an operator $\hat{A}$ be φ_n and a_n respectively: $\hat{A}\varphi_n = a_n\varphi_n$. If $f(x)$ is a function that we can expand in powers of x, show that φ_n is an eigenfunction of $f(\hat{A})$ with eigenvalue $f(a_n)$:

$$f(\hat{A})\varphi_n = f(a_n)\varphi_n$$

2. For a two-level system with an Hamiltonian $H = \begin{pmatrix} \varepsilon_a & V_{ab} \\ V_{ba} & \varepsilon_b \end{pmatrix}$

a) Show that the eigenvalues are $\varepsilon_{\pm} = \mathrm{E} \pm \sqrt{\Delta^2 + |\mathrm{V_{ab}}|^2}$

where $\Delta = \dfrac{\varepsilon_a - \varepsilon_b}{2}$ and $\mathrm{E} = \dfrac{\varepsilon_a + \varepsilon_b}{2}$.

b) If we define a transformation $\tan 2\theta = \dfrac{|V_{ab}|}{\Delta}$, find the form of the eigenvectors of the coupled states $|\varphi_+\rangle$, $|\varphi_-\rangle$. What is the similarity transformation that takes you from the $\{|\varphi_+\rangle, |\varphi_-\rangle\}$ to the $\{|\varphi_a\rangle, |\varphi_b\rangle\}$ basis? Is this operator unitary?

c) Verify that this basis is normalized and orthogonal.

3. Convince yourself that

$$\exp(iG\lambda)A\,\exp(-iG\lambda) = A + i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] + \ldots + \left(\frac{i^n\lambda^n}{n!}\right)\left[G,\left[G,\left[G\ldots[G,A]\right]\right]\ldots\right] + \ldots$$

where G is a Hermetian operator and λ is a real parameter.

4. Just as $U(t,t_0) = \exp[-iH\,t/\hbar]$ is the time-evolution operator which displaces $\psi(\bar{r},t)$ in time,

$$D(\bar{r},\bar{r}_0) = \exp\left(-i\frac{\bar{p}}{\hbar}\cdot(\bar{r}-\bar{r}_0)\right)$$

is the spatial displacement operator that moves ψ in space.

a) Defining $D(\bar{\lambda}) = \exp\left(-i\frac{\bar{p}}{\hbar}\cdot\bar{\lambda}\right)$, show that the transformation

$$D^{\dagger} r D = r + \lambda$$

where λ is a displacement vector. The relationship in Problem 3 will be useful here.

b) Show that the wavefunction of the state

$$|\phi\rangle = D|\psi\rangle$$

is the same as the wavefunction of the state $|\psi\rangle$, only shifted a distance λ. Write out $\phi(x) = \langle x|\phi\rangle$ explicitly if $|\phi\rangle$ is the ground state of the one-dimensional harmonic oscillator.

5. The Hamiltonian for a degenerate two-level system is

$$H_o = |a\rangle\, \varepsilon_0\, \langle a| + |b\rangle\, \varepsilon_0\, \langle b|$$

At time t = 0 a perturbation is applied:

$$V(t) = |a\rangle\, V_{ba}(t)\, \langle b| + |b\rangle\, V_{ab}(t)\, \langle a|$$

where $V_{ab}(t) = V_{ba}(t)^* = V(1 - \exp(-\gamma t))$.

a) Does the Hamiltonian commute at all times?

b) If the system is initially in prepared in state $|b\rangle$ ($t \le 0$), what is the state of the system for $t > 0$?

c) What is the probability of finding the system in $|a\rangle$ for $t > 0$?

d) Describe the behavior of this system in the limits $\gamma \to 0$ and $\gamma \to \infty$.

6. Time-Development of the Density Matrix

(a) Using the time-dependent Schrödinger equation, show that the time-dependence of the density matrix $\rho = |\psi\rangle\langle\psi|$ is given by the Liouville-Von Neumann equation:

$$\frac{\partial \rho}{\partial t} = \frac{-i}{\hbar}[H, \rho]$$

(b) Show that the time dependence of ρ obtained by directly integrating the Liouville-Von Neumann equation from 0 to t is the same as $\rho(t) = U\rho(0)U^{\dagger}$.