# 1.022 Introduction to Network Models

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Lectures 13 and 14

# Outline

Dynamical systems:

- Linear and non-linear
- Convergence
- Linear algebra and Lyapunov functions
- discrete and continuous

# Dynamical systems

- Discrete time system: time indexed by $k$
  - let $x(k) \in \mathbb{R}^n$ denote system state
  - examples: state of infection, levels of consumption for a product, opinions
  - amount of labor, steele and coal available in an economy, ...

- System dynamics: for any $k \geq 0$

$$x(k+1) = F(x(k)) \tag{1}$$

for some $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

- Primary questions:
  - Is there an equilibrium $x^\star \in \mathbb{R}^n$, i.e. $x^\star = F(x^\star)$.
  - If so, does $x(k) \rightarrow x^\star$ and how quickly?

# Linear dynamical systems

- Linear system dynamics: for any $k \geq 0$

$$x(k+1) = Ax(k) + b \tag{2}$$

  - for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$
  - example: Leontif's input-output model of economy: output from one industrial sector may become an input to another industrial sector.
  - best response to the consumption level of friends

- We'll study
  - Existence and characterization of equilibrium.
  - Convergence.

- Initially, we'll consider $b = \mathbf{0}$
  - Later, we shall consider generic $b \in \mathbb{R}^n$

# Linear dynamical systems

- Consider

$$\begin{aligned} x(k) &= Ax(k-1) \\ &= A \times Ax(k-2) \\ &\cdots \\ &= A^k x(0) \end{aligned}$$

- So what is $A^k$?

- For $n = 1$, let $A = a \in \mathbb{R}_+$:

$$x(k) = a^k x(0) \overset{k \to \infty}{\to} \begin{cases} 0 \text{ if } 0 \leq a < 1 \\ x(0) \text{ if } a = 1 \\ \infty \text{ if } 1 < a. \end{cases}$$

# Linear dynamical systems

- For $n > 1$, if $A$ were diagonal, i.e.,

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

  - Then

$$A^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}$$

  - and, likely that we can analyze behavior $x(k)$
  - but, most matrices are not diagonal

# Linear dynamical systems

- Diagonalization: for a large class of matrices $A$,
  - it can be represented as $A = S\Lambda S^{-1}$, where diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

  - and $S \in \mathbb{R}^{n\times n}$ is invertible matrix

- Then

$$\begin{aligned} x(k) &= (S\Lambda S^{-1})^k x(0) \\ &= S\Lambda^k S^{-1} x(0) \;=\; S\Lambda^k c \end{aligned}$$

where $c = c(x(0)) = S^{-1}x(0) \in \mathbb{R}^n$

# Linear dynamical systems

- Suppose

$$S = \begin{pmatrix} | & & | \\ s_1 & \dots & s_n \\ | & & | \end{pmatrix}$$

- Then

$$\begin{aligned} x(k) &= S\Lambda^k c \\ &= \sum_{i=1}^{n} c_i \lambda_i^k s_i \end{aligned}$$

- Let $0 \leq |\lambda_n| \leq |\lambda_{n-1}| \leq \cdots \leq |\lambda_2| < |\lambda_1|$

$$x(k) = \sum_{i=1}^{n} c_i \lambda_i^k s_i \;=\; \lambda_1^k \Big( c_1 s_1 + \sum_{i=2}^{n} c_i \big(\frac{\lambda_i}{\lambda_1}\big)^k s_i \Big)$$

- Then

$$\|x(k)\| \overset{k\to\infty}{\to} \begin{cases} 0 \text{ if } |\lambda_1| < 1 \\ |c_1| \|s_1\| \text{ if } |\lambda_1| = 1 \\ \infty \text{ if } |\lambda_1| > 1 \end{cases}$$

- moreover, for $|\lambda_1| > 1$,

$$\|\lambda_1^{-k} x(k) - c_1 s_1\| \to 0.$$

- When can a matrix $A \in \mathbb{R}^{n \times n}$ be diagonalized?
  - 
  - When $A$ has $n$ distinct eigenvalues, for example
  - Another example: Real symmetric matrices
  - In general, all matrices are block-diagonalizable a la Jordan form

- Eigenvalues of $A$
  - Roots of $n$ order (characteristic) polynomial: $\det(A - \lambda I) = 0$
  - Let them be $\lambda_1, \ldots, \lambda_n$

- Eigenvectors of $A$
  - Given $\lambda_i$, let $s_i \neq \mathbf{0}$ be such that $As_i = \lambda_i s_i$
  - Then $s_i$ is eigenvector corresponding to eigenvalue $\lambda_i$

- If all eigenvalues are distinct, then eigenvectors are linearly independent.

# Diagonalization

- If all eigenvalues are distinct, then eigenvectors are linearly independent.
- **Proof.** Suppose not and let $s_1$, $s_2$ are linearly dependent.
  - 
    - that is, $a_1 s_1 + a_2 s_2 = \mathbf{0}$ for some $a_1, a_2 \neq 0$
    - that is, $a_1 A s_1 + a_2 A s_2 = \mathbf{0}$, and hence $a_1 \lambda_1 s_1 + a_2 \lambda_2 s_2 = \mathbf{0}$
    - multiplying first equation by $\lambda_2$ and subtracting second

$$a_1(\lambda_2 - \lambda_1)s_1 = \mathbf{0}$$

    - that is, $a_1 = 0$; similarly, $a_2 = 0$. Contradiction.
    - argument can be similarly extended for case of $n$ vectors.

# Diagonalization

- If all eigenvalues are distinct ($\lambda_i \neq \lambda_j,\ i \neq j$), then eigenvectors, $s_1, \ldots, s_n$, are linearly independent.

- Therefore, we have invertible matrix $S$, where

$$S = \begin{pmatrix} | & & | \\ s_1 & \ldots & s_n \\ | & & | \end{pmatrix}$$

- Consider diagonal matrix of eigenvalues

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

- Consider

$$\begin{aligned} AS &= \begin{pmatrix} | & & | \\ \lambda_1 s_1 & \dots & \lambda_n s_n \\ | & & | \end{pmatrix} \\ &= \begin{pmatrix} | & & | \\ s_1 & \dots & s_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \\ &= S\Lambda \end{aligned}$$

- Therefore, we have diagonalization $A = S\Lambda S^{-1}$

- Remember: not every matrix is diagonalizable, e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- Let us consider linear system with $b \neq \mathbf{0}$:

$$\begin{aligned} x(k+1) &= Ax(k) + b \\ &= A(Ax(k-1)+b)+b \;=\; A^2x(k-1) + (A+I)b \\ &\dots \\ &= A^k x(0) + \Big(\sum_{j=0}^{k-1} A^{k-j-1}\Big) b. \end{aligned}$$

- Let $A = S\Lambda S^{-1}$, $c = S^{-1}x(0)$ and $d = S^{-1}b$. Then

$$x(k+1) = \sum_{i=1}^{n} c_i s_i \lambda_i^k + d_i s_i (\sum_{j=0}^{k-1} \lambda_i^j)$$

- Let $A = S\Lambda S^{-1}$, $c = S^{-1}x(0)$ and $d = S^{-1}b$. Then

$$x(k+1) = \sum_{i=1}^{n} c_i s_i \lambda_i^k + d_i s_i (\sum_{j=0}^{k-1} \lambda_i^j)$$

- Let $0 \le |\lambda_n| \le |\lambda_{n-1}| \le \cdots \le |\lambda_2| \le |\lambda_1|$. Then
  - If $|\lambda_1| \ge 1$, the sequence is divergent ($\to \infty$)
  - If $|\lambda_1| < 1$, it converges as

$$x(k) \overset{k\to\infty}{\to} \sum_{i=1}^{n} s_i \frac{d_i}{1-\lambda_i}$$
$$= S \begin{pmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_n} \end{pmatrix} S^{-1} b \; = \; (I - A)^{-1} b$$

- For linear system, equilibrium $x^\star$ should satisfy

$$x^\star = Ax^\star + b$$

- The solution to the above exists when $A$ does not have an eigenvalue equal to 1, which is

$$x^\star = (I - A)^{-1} b$$

  - But, as discussed, it may not be reached unless $|\lambda_1| < 1$! (unstable equilibrium)

# Nonlinear dynamical systems

- Consider nonlinear system

$$\begin{aligned} x(k+1) &= F(x(k)) \\ &= x(k) + (F(x(k)) - x(k)) \\ &= x(k) + G(x(k)) \end{aligned}$$

where $G(x) = F(x) - x$

- Continuous approximation of the above (replace $k$ by time index $t$)

$$\frac{dx(t)}{dt} = G(x(t))$$

- When does $x(t) \to x^\star$?

# Lyapunov function

- Let there be a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$
- Such that
  1. $V$ is minimum at $x^\star$
  2. $\frac{dV(x(t))}{dt} < 0$ if $x(t) \neq x^\star$

     that is, $\nabla V(x(t))^T G(x(t)) < 0$ if $x(t) \neq x^\star$
- Then $x(t) \to x^\star$

# Lyapunov function: An Example

- A simple model of Epidemic
  - Let $I(k) \in [0, 1]$ be fraction of population that is infected
  - and $S(k) \in [0, 1]$ be the fraction of population that is susceptible to infection
  - Population is either infected or susceptible: $I(k) + S(k) = 1$

- Due to "social interaction" they evolve as

$$I(k+1) = I(k) + \beta I(k)S(k)$$
$$S(k+1) = S(k) - \beta I(k)S(k)$$

where $\beta \in (0, 1)$ is a parameter captures "infectiousness"

- Question: what is the equilibrium of such a society?

# Lyapunov function: An Example

- Since $I(k) + S(k) = 1$, we can focus only on one of them, say $S(k)$
- Then

$$S(k+1) = S(k) - \beta(1 - S(k))S(k)$$

- That is, continuous approximation suggests

$$\frac{dS(t)}{dt} = -\beta(1 - S(t))S(t).$$

- An easy Lyapunov function is $V(S) = S$

- For $V(S) = S$:

$$\frac{dV(S(t))}{dt} = V'(S(t))\frac{dS(t)}{dt} = -\beta(1 - S(t))S(t)$$

- Then, for $S(t) \in [0, 1)$ if $S(t) \neq 0$,

$$\frac{dV(S(t))}{dt} < 0$$

- And $V$ is minimized at 0

- Therefore, if $S(0) < 1$, then $S(t) \to 0$: entire population is *infected*!