HW#2-SOLUTION: Stress & Strength – Nano-Indentation

October 6, 2003

MIT – 1.033/1.57
Fall 2003
Instructor: Franz-Josef ULM

---

1 Statically Admissible Stress Fields

A statically admissible stress field is a stress field which satisfies (*i*) the force boundary conditions, (*ii*) the stress vector continuity condition on any surface in the material; (*iii*) the symmetry of the stress tensor; (*iv*) the momentum balance.

1.1 Boundary Conditions

For the nanoindentation test, the boundary conditions are:

- In Ω_1:
  - Frictionless contact:

$$\text{on } z=0; r<r_0; \mathbf{n}=-\mathbf{e}_z : \mathbf{T}(\mathbf{n}=-\mathbf{e}_z)\cdot\mathbf{t}=0 \leftrightarrow \left\{ \begin{array}{c} \mathbf{t}=\mathbf{e}_r : \sigma_{rz}=0 \\ \mathbf{t}=\mathbf{e}_\theta : \sigma_{\theta z}=0 \end{array} \right\} \quad (1)$$

  - Force boundary condition:

$$\text{on } z=0; r<r_0; \mathbf{n}=-\mathbf{e}_z : \mathcal{N}^d = F\mathbf{e}_z \equiv \int_{A=\pi r_0^2} \boldsymbol{\sigma}\cdot\mathbf{n}da = -\int_{A=\pi r_0^2} \boldsymbol{\sigma}\cdot\mathbf{e}_z da \quad (2)$$

  - From a combination of (1) and (2), it follows:

$$\text{on } z=0; r\leq r_0; \mathbf{n}=-\mathbf{e}_z : \left\{ \begin{array}{c} \langle\sigma_{rz}\rangle_A = 0 \\ \langle\sigma_{\theta z}\rangle_A = 0 \\ \langle\sigma_{zz}\rangle_A = -H \end{array} \right\} \quad (3)$$

  where $\langle\sigma_{ij}\rangle_A = \frac{1}{A}\int_{A=\pi r_0^2}\sigma_{ij}da$ stands for the stress average of σ_{ij} over the surface $A=\pi r_0^2$, and $H=F/A$ is the micro-hardness measured in the nanoindentation test. Note that it cannot *a priori* be concluded that $\sigma_{zz}=-F/A$, since σ_{zz} may not be constant over the contact area.

- In Ω_2:

$$\text{on } z=0; r>r_0; \mathbf{n}=-\mathbf{e}_z : \mathbf{T}(\mathbf{n}=-\mathbf{e}_z)=0 \Rightarrow \left\{ \begin{array}{c} \sigma_{rz}=0 \\ \sigma_{\theta z}=0 \\ \sigma_{zz}=0 \end{array} \right\} \quad (4)$$

1.2 Continuity of Stress Vector

On the interface between domain Ω_1 and Ω_2:

$$z > 0; r = r_0; \mathbf{n} = \mathbf{e}_r : \mathbf{T}^{(1)}(\mathbf{n} = \mathbf{e}_r) + \mathbf{T}^{(2)}(\mathbf{n} = -\mathbf{e}_r) = 0 \Rightarrow \left\{ \begin{array}{c} \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} \\ \sigma_{\theta r}^{(1)} = \sigma_{\theta r}^{(2)} \\ \sigma_{zr}^{(1)} = \sigma_{zr}^{(2)} \end{array} \right\} \tag{5}$$

1.3 Form of the Stress Tensor

Given the rotational symmetry of the problem, $\sigma_{\theta r} = 0$ in Ω. The stress tensor, therefore, is of the diagonal form:

$$\text{in } \Omega : \boldsymbol{\sigma} = \sigma_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{6}$$

which satisfies the symmetry condition, $\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma}$, and the boundary conditions (3) and (4), and the continuity condition (5).

1.4 Momentum Balance

Neglecting body forces, the stress tensor $\boldsymbol{\sigma}$ in Ω must satisfy the following momentum balance equations (cylinder coordinates):

$$\text{in } \Omega : \text{div}\, \boldsymbol{\sigma} = 0 : \left\{ \begin{array}{c} \mathbf{e}_r : \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}[\sigma_{rr} - \sigma_{\theta\theta}] = 0 \\ \mathbf{e}_\theta : \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \\ \mathbf{e}_z : \frac{\partial \sigma_{zz}}{\partial z} = 0 \end{array} \right\} \tag{7}$$

1.5 Application

- In Ω_1:

$$\text{in } z > 0; r < r_0 : \sigma'_{rr} = q'; \quad \sigma'_{\theta\theta} = q''; \quad \sigma'_{zz} = \sigma \tag{8}$$

From $(7)_1$:

$$\frac{\partial \sigma_{rr}}{\partial r} = 0;\ \sigma'^{(1)}_{rr} = \sigma'^{(1)}_{\theta\theta} \Rightarrow q' = q'' \tag{9}$$

From $(7)_3$ and $(3)_3$:

$$\frac{\partial \sigma_{zz}}{\partial z} = 0;\ \sigma'^{(1)}_{zz} = \sigma = -H \tag{10}$$

- In Ω_2: The stress field,

$$\text{in } z > 0; r > r_0 : \sigma'_{rr} = -q(r_0/r)^2; \quad \sigma'_{\theta\theta} = q(r_0/r)^2 \tag{11}$$

satisfies the boundary condition (4), the momentum balance equations $(7)_1$:

$$\frac{\partial \sigma'_{rr}}{\partial r} + \frac{1}{r}[\sigma'_{rr} - \sigma'_{\theta\theta}] = 2\frac{q}{r}\left(\frac{r_0}{r}\right)^2 + \frac{q}{r}\left[-\left(\frac{r_0}{r}\right)^2 - \left(\frac{r_0}{r}\right)^2\right] = 0 \tag{12}$$

The stress continuity (5) is satisfied for:

$$z > 0; r = r_0 : \sigma'^{(1)}_{rr} = \sigma'^{(2)}_{rr} \Leftrightarrow q' \equiv -q \tag{13}$$

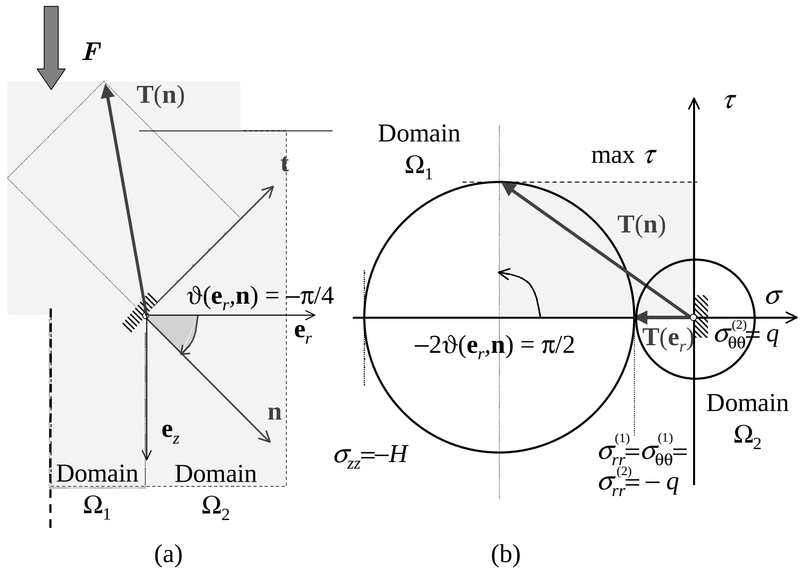

Figure 1: Stress Representation of the Nano-Indentation stress model: (a) material plane; (b) Mohr plane.

1.6 Mohr Representation

The stress field are:

$$\text{in } \Omega_1 \quad : \quad \boldsymbol{\sigma}'^{(1)} = q\left[\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta\right] - H\mathbf{e}_z \otimes \mathbf{e}_z \tag{14}$$

$$\text{in } \Omega_2 \quad : \quad \boldsymbol{\sigma}'^{(2)} = q\left(\frac{r_0}{r}\right)^2 \left[-\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta\right] \tag{15}$$

The Mohr circles are displayed in Figure 1b, for $H > q$. The maximum shear occurs in Ω_1 and the corresponding stress vector has the components:

$$\max \tau = \frac{H-q}{2}; \sigma = -\frac{H+q}{2} \tag{16}$$

The maximum shear stress occurs on the surface oriented by (see Figure 1a):

$$\mathbf{n} = \frac{\sqrt{2}}{2}\left[\mathbf{e}_r + \mathbf{e}_z\right] \tag{17}$$

2 Mohr-Coulomb Criterion

The unknown of the problem is the stress quantity q. It is determined through application of the Mohr-Coulomb criterion:

- In Ω_1:

$$\sigma_I^{(1)} = \sigma_{II}^{(1)} = -q; \sigma_{III}^{(1)} = -H \tag{18}$$

Use in the Mohr-Coulomb Criterion reads:

$$f(\boldsymbol{\sigma}'^{(1)}) = -q(1+\sin\varphi) + H(1-\sin\varphi) - 2c\cos\varphi \leq 0 \quad (19)$$

- In Ω_2:

$$\sigma_I^{(2)} = q\left(\frac{r_0}{r}\right)^2; \sigma_{II}^{(2)} = 0; \sigma_{III}^{(2)} = -q\left(\frac{r_0}{r}\right)^2 \quad (20)$$

Use in the Mohr-Coulomb criterion reads:

$$f(\boldsymbol{\sigma}) = 2q\left(\frac{r_0}{r}\right)^2 - 2c\cos\varphi \leq 0 \quad (21)$$

From (20), we obtain:

$$q \leq c\left(\frac{r}{r_0}\right)^2 \cos\varphi \Rightarrow \max_{r=r_0} q = c\cos\varphi \quad (22)$$

Use in (19) gives the sought relation between the Hardness measurement and the Mohr-Coulomb model parameters:

$$H \leq \max H = c\cos\varphi\left(\frac{3+\sin\varphi}{1-\sin\varphi}\right) \quad (23)$$

The representation of this limit state is shown in Figure 2, in both the physical plane and the Mohr-plane. The critical stress state is reached in Ω_1 on a material surface oriented by:

$$\mathbf{n} = \cos\phi\mathbf{e}_r - \sin\phi\mathbf{e}_z; \phi = -\pi/4 + \varphi/2 \quad (24)$$

3 Refined Approach

We consider that the stress state in Ω_2 was constant. From the momentum balance $(7)_1$ we find:

$$\frac{\partial\sigma_{rr}}{\partial r} = 0 \Rightarrow \sigma_{rr}^{(2)} = \sigma_{\theta\theta}^{(2)} \quad (25)$$

This stress field is statically admissible provided that the stress continuity along $r = r_0$ is ensured:

$$\sigma_{rr}^{(1)} = \sigma_{rr}^{(2)} = q' \quad (26)$$

It follows:

- In Ω_1:

$$\sigma_I^{(1)} = \sigma_{II}^{(1)} = q'; \sigma_{III}^{(1)} = -H \quad (27)$$

Use in the Mohr-Coulomb Criterion reads:

$$f(\boldsymbol{\sigma}'^{(1)}) = q'(1+\sin\varphi) + H(1-\sin\varphi) - 2c\cos\varphi \leq 0 \quad (28)$$

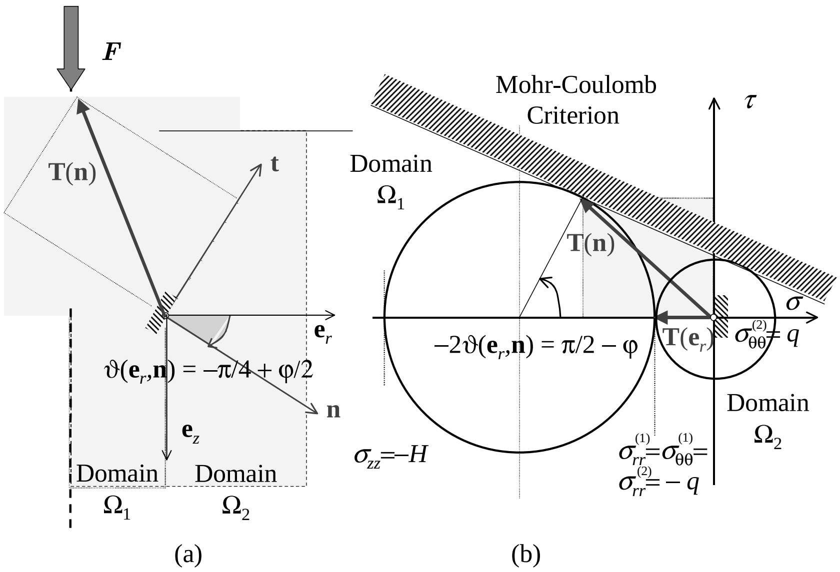

Figure 2: Stress State at strength limit: (a) material plane; (b) Mohr plane.

- In Ω_2:

$$\sigma_I^{(2)} = 0; \sigma_{II}^{(2)} =_{III}^{(2)}= q' \tag{29}$$

Use in the Mohr-Coulomb criterion yields:

$$f(\boldsymbol{\sigma}'^{(2)}) = -q'(1 - \sin\varphi) - 2c\cos\varphi \leq 0 \tag{30}$$

From (30) we obtain:

$$-q' \leq \frac{2c\cos\varphi}{1 - \sin\varphi} \tag{31}$$

Use in (28) yields:

$$f(\boldsymbol{\sigma}'^{(1)}) \leq 0 : H \leq \max H = 2c\cos\varphi \left[\frac{1}{1 - \sin\varphi} + \frac{1 + \sin\varphi}{(1 - \sin\varphi)^2}\right] = \frac{4c\cos\varphi}{(1 - \sin\varphi)^2} \tag{32}$$

This improves stress-strength solution is displayed in the Mohr-Plane in Figure 4. It is obtained by shifting the Mohr-circle in domain Ω_2 along the normal stress axis into the compression domain. This yields a higher micro-hardness than the first solution (see Figure 3). Since the stress field is statically admissible and since it satisfies the strength criterion, this higher value is closer to the 'real' micro-hardness of the material. The material planes along which the material realizes the strength criterion are still the same as before (see Figure 2a), but extends now also in domain Ω_2. This is displayed in Figure 5 (NOT ASKED).

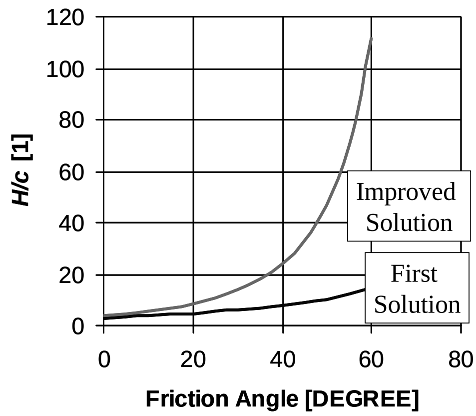

Figure 3: Comparison of the normalized micro-hardness values versus friction angle.

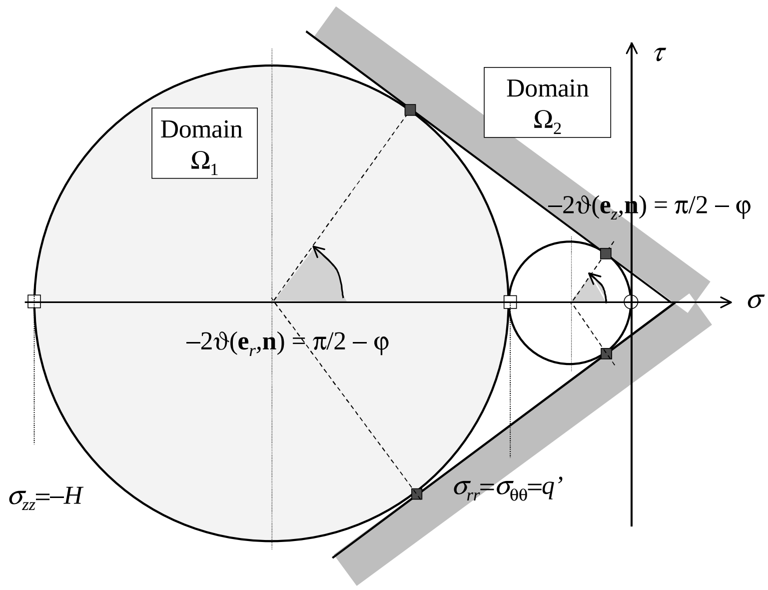

Figure 4: Improved Stress-Strength Solution in the Mohr-Plane. The stress state is statically admissible, and satisfies the Mohr-Coulomb strength criterion.

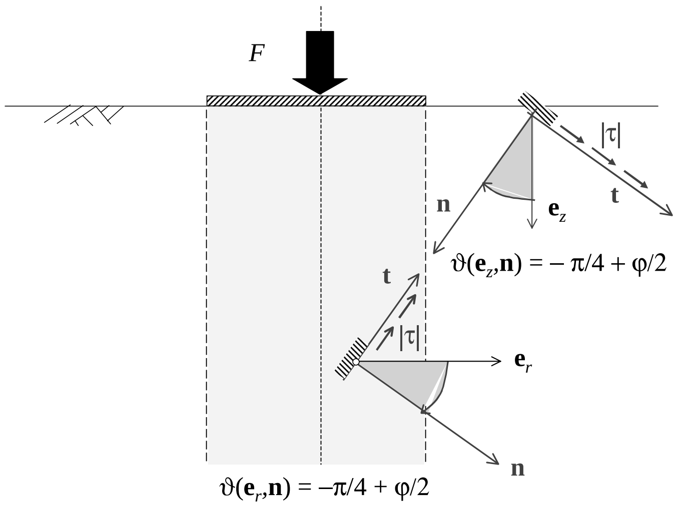

Figure 5: Improved Stress-Strength solution: Display of normal planes on which the strength criterion is achieved.