

# 1.033/1.57 H#3: Elasticity & Elasticity Bounds

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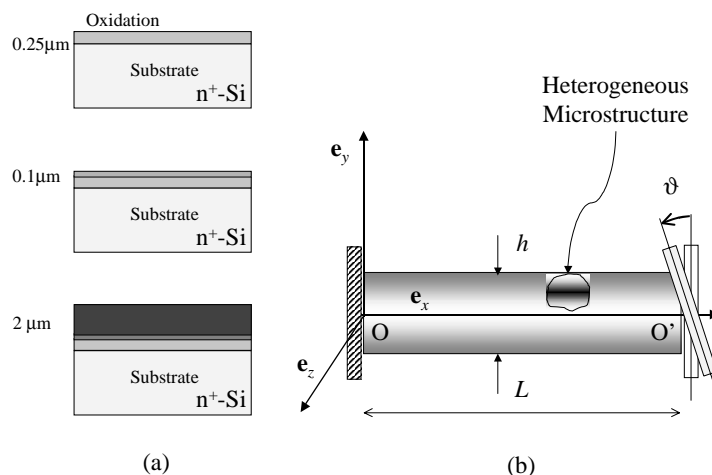
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**Elasticity Bounds of Microflexural Structures (MEMS TYPE)** Microelectromechanical systems (MEMS) are miniature electromechanical sensor and actuator that are used as accelerometers, micro-pumps, micro-turbines, and optical switches. The majority of MEMS devices are based on the movement of cantilever beams or thin membranes. MEMS are often made of thin layers of polysilicon, amorphous silicon, silica, oxides or polyimides, and the material is deposited in layers on some substratum (see figure (a) below). We want to give a lower bound and an upper bound for the apparent elasticity modulus of such microflexural structures.

To this end, we consider a cantilever beam of length  $L$  along the  $x$ -axis and of height  $h$  and width  $b$ , as shown in figure (b) below. At  $x = 0$ , the section (centered around  $O$ ) is in contact without friction with a fixed rigid plate. At the other end, at  $x = L$ , the section (centered around  $O'$ ) fixed on a frictionless rigid plate is subjected to a rotation of intensity  $\vartheta \ll 1$  around the  $z$ -axis, such that:

$$\text{at } x = L : \xi_x^d = [\mathbf{O}'\mathbf{M} \times \vartheta \mathbf{e}_z] \cdot \mathbf{e}_x = -\vartheta y$$

The surface of the beam is stress free. In a very first approximation, we consider that the heterogeneous matter constituting the beam is a linear isotropic elastic material defined by Young's Modulus  $E(\mathbf{x})$  and Poisson's ratio  $\nu(\mathbf{x})$ .



Problem set: Microflexural system: (a) Layered deposition manufacturing; (b) Cantilever structure.

1. **Equivalent Homogeneous Sample:** Let  $E_{app}$  be the apparent elastic modulus of the equivalent homogeneous sample. We consider a uniaxial stress field of the form,

$$\text{in } \Omega : \boldsymbol{\sigma} = \sigma(y)\mathbf{e}_x \otimes \mathbf{e}_x$$

- (a) Determine  $\sigma(y)$ , using the isotropic linear elastic material law ( $E = E_{app} = \text{const}$ ), and show that the bending moment at  $x = L$  obeys to the section-type constitutive law:

$$M_z = E_{app} I_z \frac{\vartheta}{L}$$

where  $I_z$  is the bending inertia moment around the  $z$ -axis.

- (b) Determine the corresponding displacement solution  $\boldsymbol{\xi}$ .
- (c) Specify briefly why  $(\boldsymbol{\xi}, \boldsymbol{\sigma})$  is the solution of the equivalent homogeneous bending problem.
2. **Heterogeneous Sample:** We now consider the heterogeneous bending sample (i.e.,  $E = E(\mathbf{x}), \nu = \nu(\mathbf{x})$ ).

- (a) **Target Solution:** Let  $(\boldsymbol{\xi}, \boldsymbol{\sigma})$  be the (exact) displacement and stress solution in the heterogeneous sample subjected to the rotation at  $x = L$ . For this solution, show that the elastic energy stored per unit length in the beam reads:

$$\frac{W(\boldsymbol{\xi})}{L} = \frac{W^*(\boldsymbol{\sigma})}{L} = \frac{1}{2} M_z(x = L) \frac{\vartheta}{L}$$

- (b) **Lower Bound:** Inspired by the elastic homogeneous solution developed above, we consider for the heterogeneous sample, an approximated stress field of the form:

$$\text{in } \Omega : \boldsymbol{\sigma}' = \sigma' \frac{y}{L} \mathbf{e}_x \otimes \mathbf{e}_x$$

Determine a lower bound of the apparent  $E$ -modulus of the heterogeneous sample.

- (c) **Upper Bound:** Inspired by the elastic homogeneous solution developed above, we consider for the heterogeneous sample a displacement field of the form:

$$\text{in } \Omega : \boldsymbol{\xi}' = -\vartheta y \frac{x}{L} \mathbf{e}_x + \frac{\vartheta}{2L} (x^2 + \alpha(y^2 - z^2)) \mathbf{e}_y + \alpha \frac{\vartheta}{L} y z \mathbf{e}_z$$

Determine an upper bound of the apparent  $E$ -modulus of the heterogeneous sample.

3. **Application to a Two-Layer Beam:** We consider that the beam is composed of two layers of thickness  $h_1$  and  $h_2 = h - h_1$ .  $E_1$  and  $E_2$  are the elasticity moduli of layer 1 and layer 2, respectively. The Poisson's ratio in both layers is assumed the same. Determine the lower and upper bound of the apparent elasticity modulus  $E_{app}$  of the bending beam. Display the results as a function of the relative thickness  $c = h_1/h$  for a value of  $E_1/E_2 = 2$  (corresponding, e.g. to a Quartz -layer  $\text{SiO}_2$  of  $E_1 = 380$  GPa obtained by thermal oxidation of a silicon substratum of  $E_2 = 190$  GPa). Show that one of the two bounds corresponds precisely to the considered geometrical situation.