LECTURE # 29

1.060 ENGINEERING MECHANICS II

Let us review the tools we have developed for Free Surface Flow.

1) Uniform, Steady Flow (Lectures 23 & 24)

$$Q = VA = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} \sqrt{S_o} \quad \text{(Manning's Eq.)}$$

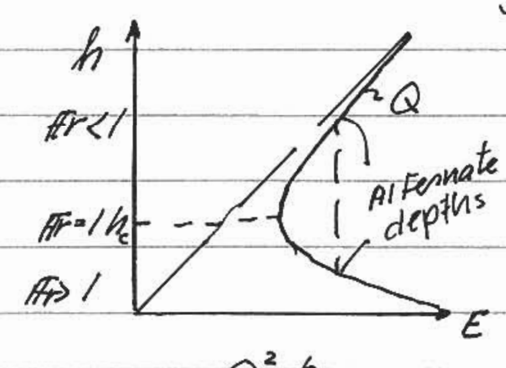
The solution determines:

h_n = Normal Depth

2) Short Transitions of Converging Flow (Lec. 25 & 26)

$$E = H - z_o = h + \frac{Q^2}{2gA^2} = \text{Specific Head}$$

$$H_1 - H_2 = \Delta H \approx 0 \Rightarrow E_1 = E_2 + (z_{o1} - z_{o2})$$



Energy Principle

$$Fr^2 = \frac{Q^2 b_s}{gA^3} = 1 = \text{Critical Flow}$$

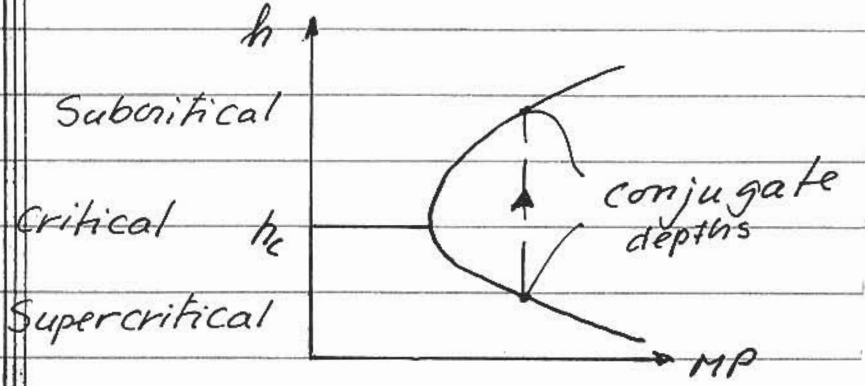
The solution determines.

h_c = Critical Depth

3) Short Transitions of Diverging Flow (Lec. 27 & 28)

$$MP = (\rho V^2 + p_{CG})A = \text{Thrust}$$

$$MP_1 = MP_2 + \sum(\text{All other Forces on Fluid})$$



Momentum Principle

Short Transitions (2 & 3) represent
RAPIDLY VARYING FLOWS
since the flow characteristics change over a short distance.

Normal Flow (1) represents a
UNIFORM (NON-VARYING) FLOW

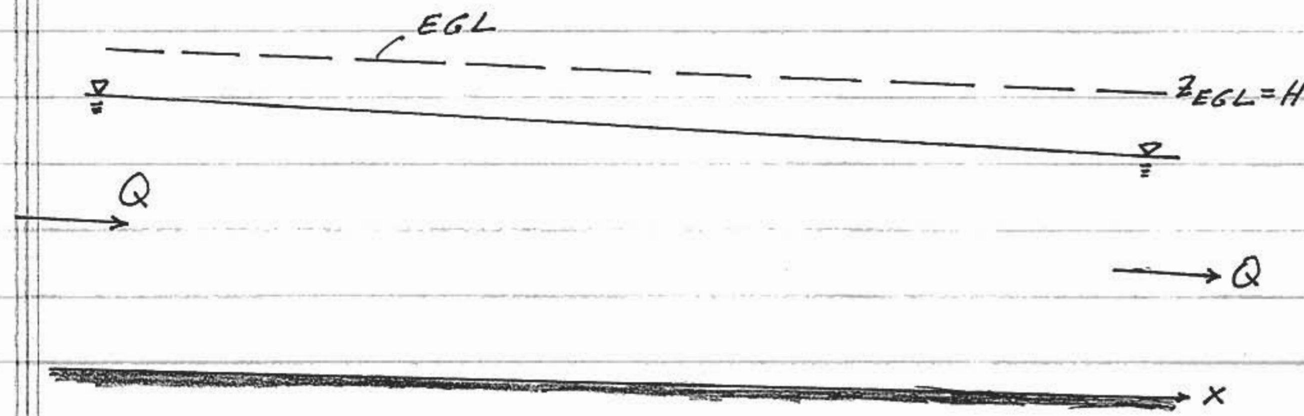
Thus, we have the tools to deal with extreme flow conditions that either vary rapidly or not at all (in space). We need to fill out the intermediate space, i.e. we need tools to analyze
GRADUALLY VARYING FLOW

Let us illustrate this "need" by an example of a flow in a prismatic channel (of course) in which there is a sudden change in the bottom slope. In the upstream channel with slope S_{o1} we have a normal depth h_{n1}, whereas the normal depth in the downstream channel is h_{n2} corresponding to a slope $S_{o2} \neq S_{o1}$. The discharge Q is the same in the two channels, so the flow is steady. The change in slope is "sudden", i.e. occurs over a short distance.

Far upstream and downstream of the change in bottom slope the flow must be uniform, since this is the only flow that can exist in an "infinitely long" channel. But if this were the case all the way to the sudden change in slope, then the mismatch in depths ($h_{n1} \neq h_{n2}$) could only be handled by out "rapidly varying flow" tools if h_{n1} and h_{n2} were either ALTERNATE or CONJUGATE DEPTHS.

Somehow the upstream and downstream flow must adjust to meet each other at the transition in slope; i.e., vary gradually in space.

GRADUALLY (OR SLOWLY) VARYING FLOW



"Gradually" or "slowly" refers to the spatial variation, i.e. the flow is still assumed to be steady. Therefore, with $\partial/\partial t = 0$

$$Q = VA = \text{Constant.}$$

However the flow characteristics such as depth, h, area, A, and Velocity, $V = Q/A$, are allowed to vary "gradually" (or "slowly") along the channel, i.e. vary slowly in x-direction. Locally, the flow behaves as a "well behaved" flow since the "gradual" variation assures that streamlines are nearly straight and nearly parallel.

Thus, we have that ($\cos\beta \approx 1$)

$$H = \text{Total Head} = \frac{V^2}{2g} + h + z_0 = \frac{Q^2}{2gA^2} + h + z_0$$

where A and h, and therefore H, are slowly varying functions of x.

Restricting the channel to be <u>prismatic</u> so that $A = A(h)$, we then have

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial h}\left(\frac{Q^2}{2gA^2} + h\right)\frac{\partial h}{\partial x} + \frac{\partial z_0}{\partial x} = \left(\frac{\partial}{\partial h}\left(\frac{Q^2}{2gA^2}\right) + 1\right)\frac{\partial h}{\partial x} - S_0$$

where

$$\partial z_0 / \partial x = -\sin\beta = -S_0$$

so S_0 is the bottom slope

In Lecture 25, when determining the minimum value of the Specific Head, we obtained

$$\frac{\partial}{\partial h}\left(\frac{Q^2}{2gA^2}\right) = -\frac{Q^2}{gA^3}\frac{\partial A}{\partial h} = -\frac{Q^2 b_s}{gA^3} = -\frac{V^2}{g h_m} = -Fr^2$$

Introducing these expressions

$$\frac{\partial H}{\partial x} = -S_f = \left(1 - Fr^2\right)\frac{\partial h}{\partial x} - S_0$$

where S_f = slope of the EGL (which must always be positive since it represents the rate of dissipation of mechanical energy).

Rearranging the terms, we obtain an equation for the slope of the free surface:

$$\frac{\partial h}{\partial x} = \frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2}.$$

$$H_2 = H_1 + \frac{\partial H_f}{\partial x}(x_2 - x_1)\ ;\ x_2 - x_1 = \text{unit length}$$

$$H_1 - H_2 = -\frac{\partial H_f}{\partial x}$$

$$-\frac{\partial H_f}{\partial x}\rho g Q = \dot{E}_{diss} = \tau_s (P \cdot 1) V$$

$$-\frac{\partial H_f}{\partial x} = \underline{S_f} = \frac{\tau_s P V}{\rho g Q} = \frac{\tau_s}{\underline{\rho g (A/P)}}$$

Darcy-Weisbach : $\tau_s = \frac{f}{8}\rho Q^2/A^2$

$$\underline{S_f = \frac{f}{8g}\frac{Q^2}{A^3/P}}$$

Manning : $8g/f = R_h^{1/3}/n^2 = (A/P)^{1/3}/n^2$

$$\underline{S_f = \frac{n^2 Q^2}{A^{10/3}/P^{4/3}}}$$

GRADUALLY VARIED FLOW EQUATION

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2}$$

S_o = bottom slope = $\sin\beta$

S_f = slope of EGL = $\frac{n^2 Q^2}{R_h^{4/3} A^2}$

Fr^2 = square of Froude Number = $\frac{Q^2 b_s}{g A^3}$

A, P, R_h = fnts (h) = fnts $(h(x))$ = fnts (x)