RECITATION #3

FLOW NET CONSTRUCTION

Some simple rules

Solid boundaries are (special) streamlines

Interior streamlines are guided by boundaries but smoother (no kinks or sharp corners)

Streamlines can never cross

Streamlines can end only at flow areas

Region(s) where flow is uniform - straight parallel streamlines - is a good place to start (end).

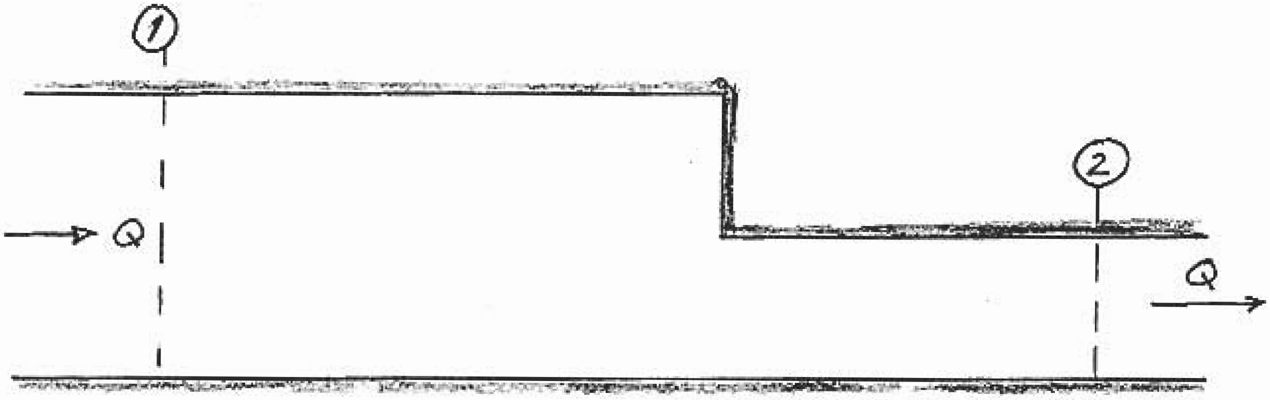
Streamlines divide flow into streamtubes

Discharge in a streamtube is constant, ΔQ.

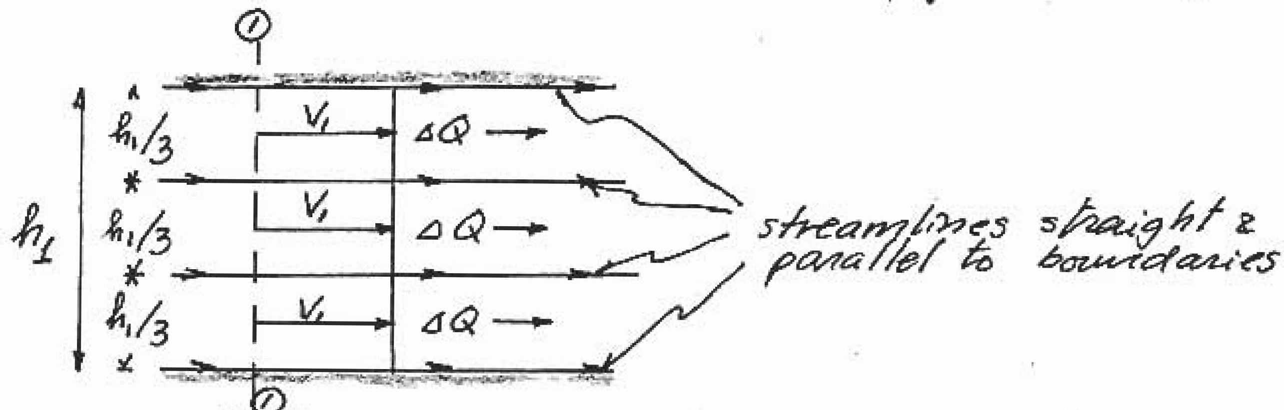
Velocity in a streamtube is $V = \Delta Q / \Delta h$ - Δh = distance between adjacent streamlines

Streamlines & Equipotential lines (h-lines) form a "square" pattern.

Example No: 1   Flow Contraction

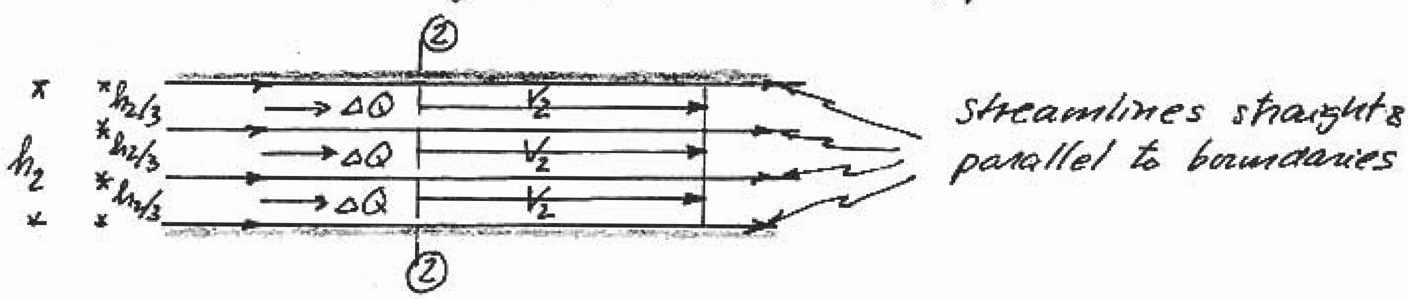


1: Far to the left of contraction, flow is uniform



$Q = h_1 V_1$; 3 streamtubes formed by 4 stream-lines: $\underline{\Delta Q} = V_1 \Delta h_1 = V_1 h_1/3 = \underline{Q/3}$

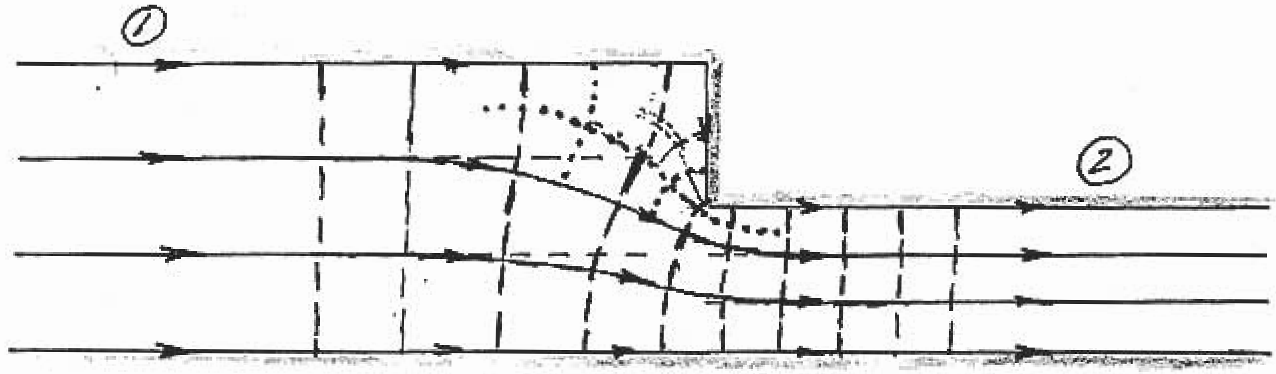
2: Far to the right of contraction, flow is uniform



$Q = h_2 V_2$; 3 streamtubes are connected to the 3 streamtubes we had at 1, so each carries a discharge ΔQ

$\frac{Q}{3} = \Delta Q = \frac{h_2}{3} V_2 = \frac{h_1}{3} V_1 \Rightarrow \frac{V_2}{V_1} = \frac{h_1/3}{h_2/3} = \frac{\Delta h_1}{\Delta h_2} = \frac{h_1}{h_2} \Rightarrow$ When Δh is small $\underline{V \text{ is large}}$

3: Flow Net for transition from ① to ②

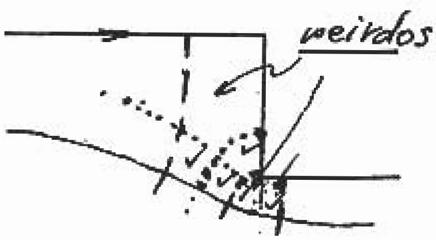


Need a smooth transition that connects the interior streamlines we obtained in regions ① and ②

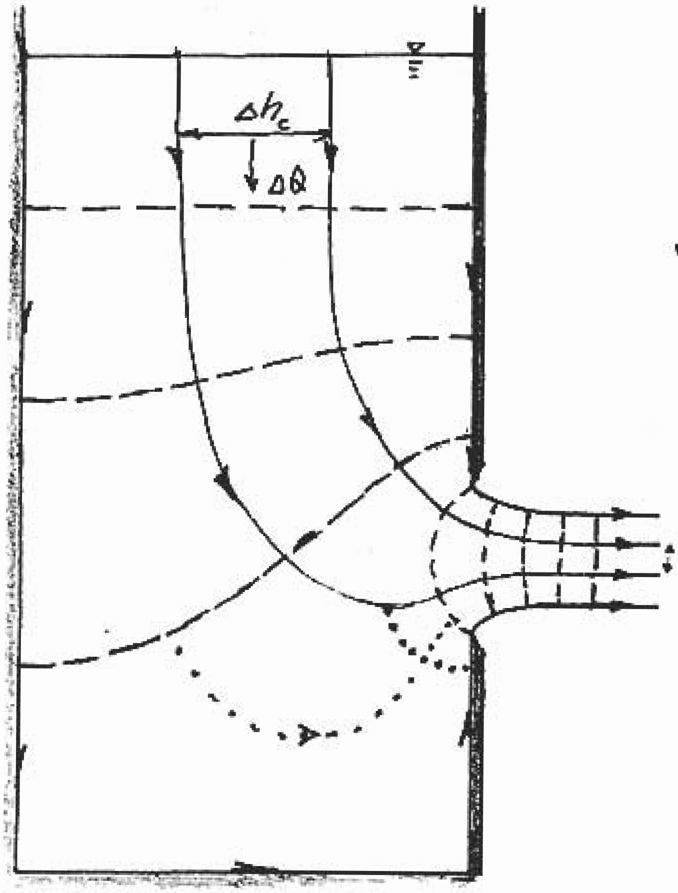
a) Start by making a first estimate of streamlines in transition zone (use a pencil - and bring an eraser!)

b) Draw the h-lines ⊥ streamlines and try to make the two families of curves form a square pattern (they probably won't at your first try, so this is where the eraser comes in).

c) Invariably, there will be some really weird "squares" around corners of the boundary streamlines. Subdivide the streamtubes in these regions e.g. into two sub-tubes. If by doing this you can get the big "weirdo-square" reduced to 3 "nice" squares and another (but smaller) weirdo, you have done pretty well.



Example No: 2 — Free Outflow from Container



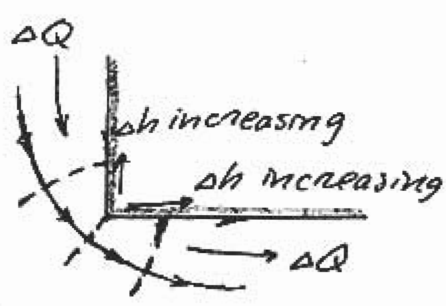
Note:

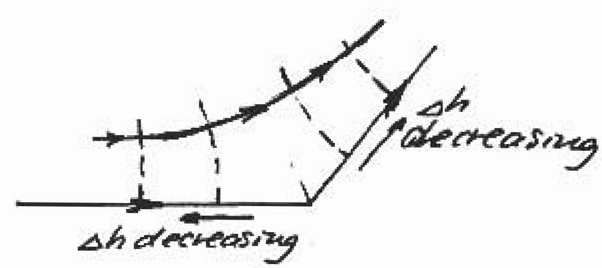
Low velocity in container far from orifice

$$\Delta Q = V_c \, \Delta h_c = V_0 \, \Delta h_0$$

$$V_c / V_0 = \Delta h_0 / \Delta h_c \ll 1$$

Example No: 3 — Flow Near Corners

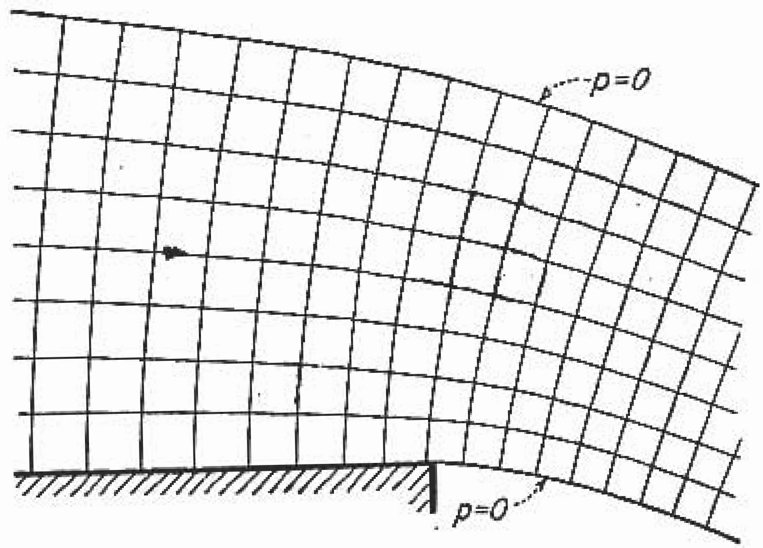




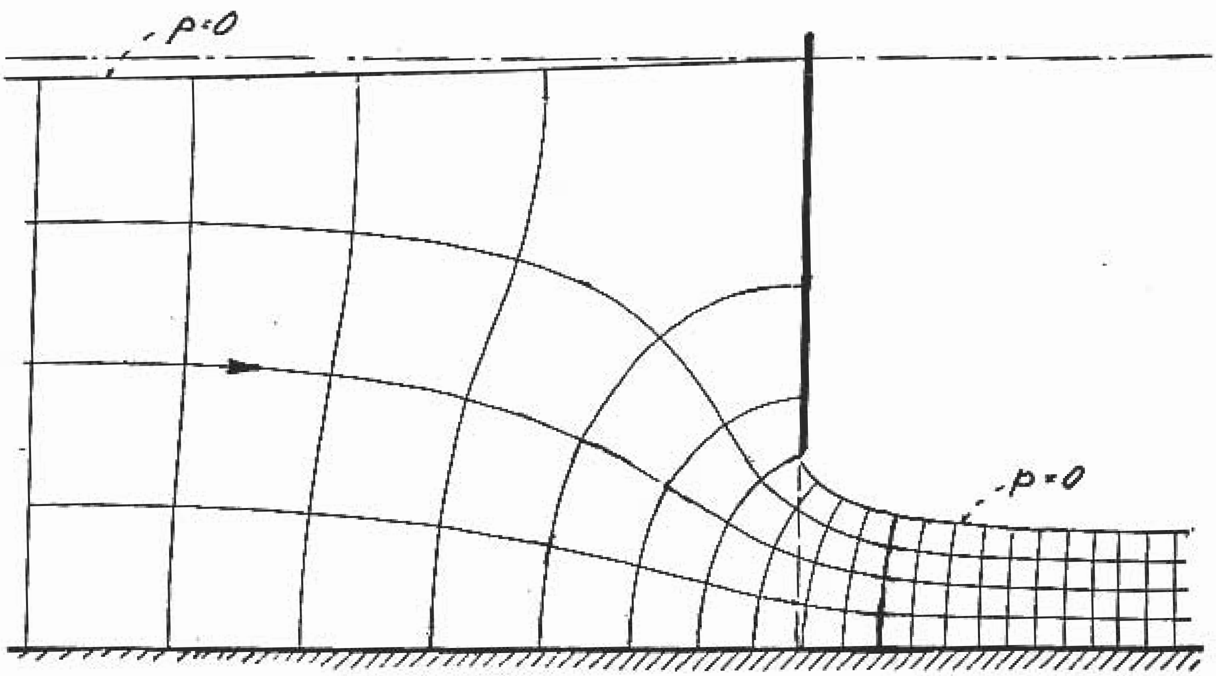
Velocity at corner pointing into flow is high and decreases away from corner

Velocity at corner pointing away from flow is low and increases away from corner.

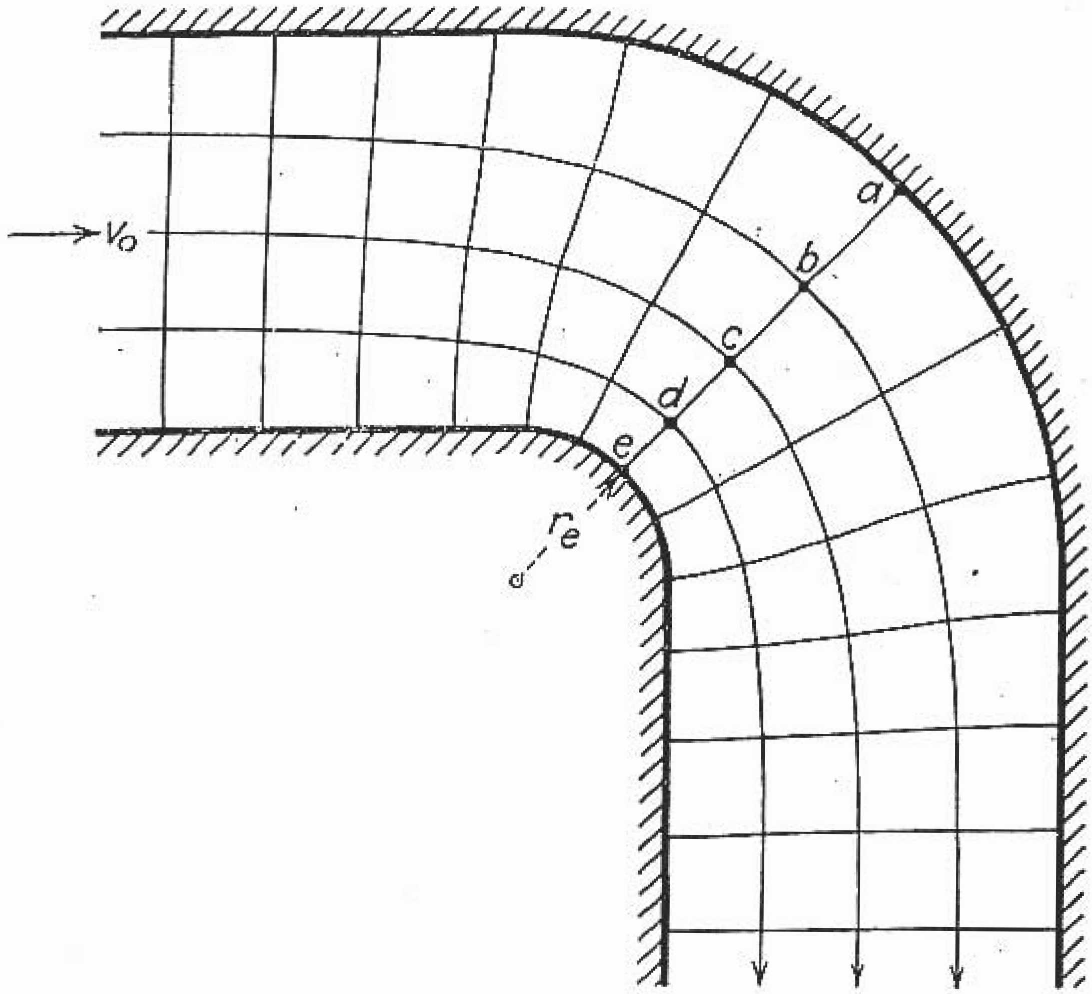
Free Surface Flow over a Brink



Free Surface Flow under a Gate



Flow around a Bend



FLOW IN POROUS MEDIA

In a saturated soil we found (Lecture #4) that the pressure in the pore fluid varies hydrostatically, if at rest, i.e.

$$p + \rho g z = \text{constant}$$

Thus, if

$$\text{grad}(p+\rho g z) = \nabla(p+\rho g z) \neq 0$$

the pore fluid must be moving

Denoting the velocity of the pore fluid (assuming the soil matrix to be fixed) by

$$\vec{q}_s = \text{seepage velocity} = (u_s, v_s, w_s)$$

We may use dimensional analysis to arrive at an equation relating $\vec{q}_s$, the dependent variable, and the independent variables (see Recitation #1)

grad $(p+\rho g z)$: the "driving force"
d : the diameter of soil particles
ρ : the pore fluid density
ν : the pore fluid kinematic viscosity.

We choose as basic units

$$led = d$$
$$med = \rho d^3$$
$$ted = d^2/\nu$$

and the nondimensional remaining independent variable is

$$\pi_1 = \frac{grad(p+\rho g z)}{\dfrac{med\,(led/ted^2)}{led^2\,led}} = \frac{grad(p+\rho g z)}{\rho \nu^2/d^3}$$

whereas the nondimensional dependent variable is

$$\pi_D = \frac{\vec{q}_s}{led/ted} = \frac{\vec{q}_s}{\nu/d}$$

Dimensional analysis then tells us that

$$\pi_D = \frac{\vec{q}_s d}{\nu} = \pi_D(\pi_1) = \text{Function of}\left(\pi_1 = \frac{grad(p+\rho g z)\, d^3}{\rho \nu^2}\right)$$

This is as far as dimensional analysis can get us. However, if we assume the "Function of" to linear, then we get

$$\pi_D = \frac{\vec{q}_s d}{\nu} = K_0 - K_1 \frac{d^3}{\rho \nu^2}\, grad(p+\rho g z)$$

Since $grad(p+\rho g z) = 0$ produces no flow, $K_0 = 0$, and we are left with

$$\vec{q}_s = -\left(K_1 \frac{d^2}{\rho \nu}\right) \text{grad}(p + \rho g z)$$

For a constant density pore fluid and g being essentially constant, this may be written in the form known as Darcy's Law

$$\vec{q}_s = \text{seepage velocity} = -K \text{ grad } H_p = -K \nabla H_p$$

where

$$H_p = \frac{p}{\rho g} + z = \text{Piezometric Head}$$

and

$$K = \text{Hydraulic Conductivity} = K_1 \frac{g d^2}{\nu} \quad [\text{velocity}]$$

With $K_1 \approx$ "constant", of the order 10^{-3} we have for water, $\nu = 10^{-6} m^2/s = 10^{-2} cm^2/s$, and $g = 9.8 m/s^2 \approx 1{,}000 cm/s^2$, a rough estimate of the Hydraulic Conductivity

$$K[cm/s] \cong [d \text{ in } mm]^2$$

Treating K as constant, and fluid & soil matrix as incompressible, we can apply continuity to the flow of pore fluid to obtain (cf. Lecture # 6)

$$\nabla \cdot \vec{q}_s = \nabla \cdot (-K \nabla H_p) = -K \nabla^2 H_p = 0$$

or

$$\nabla^2 H_p = 0$$

which is the Laplace Equation obtained also for the velocity potential for a flow of an ideal fluid (Lecture #6).

Just as done in Lecture #6 we may define streamlines and equipotential lines an show these to form a flow net. The main difference here is that the "potential" here has a physical meaning since it is the piezometric head. Construction of flow nets for 2-D flows in porous media is therefore an accurate way to determine pressure distributions and seepage velocities.

Equipotential Lines = lines of constant H_p

Streamlines $\perp$ Equipotential lines

$$q_s = -K\,(\Delta H_p / \Delta s) \quad [\text{from Flow Net}]$$

ΔQ = discharge in one stream tube $= q_s \Delta h = -K \Delta H_p \underbrace{\left(\frac{\Delta h}{\Delta s}\right)}_{\text{squares}=1}$

ΔH_p - head drop between equipotential lines = Const (ΔQ=const)

ΔH_p = same for all streamtube $\Rightarrow \Delta Q$ same for all tubes

Q = total discharge $= -K \Delta H_p \cdot N_f = -K(h_2 - h_1)\,\frac{N_f}{N_H}$

N_f = # of streamtubes, N_H = number of head drops from h_1 to h_2

RECITATION #3

Problem No: 1

1) Draw the flow net connection the uniform inflow at 1-1 and the uniform outflow at 2-2, using 4 streamtubes, on attached page

2) Determine the ratio of outflow velocity, V_2, to inflow velocity, V_1, if the depth of flow is $h_1 = 3.5\,m$ and $h_2 = 0.75\,m$.

3) If it is assumed that $p = 0$ at the top boundary of the inflow section 1-1, and also at the top of the outflow section 2-2, determine the discharge Q (m^3/s per m into paper), V_1 and V_2.

4) Estimate the pressure at the corner C

5) Estimate the pressure on the horizontal bottom immediately below the vertical wall, B.

Problem No: 2

Sketch the flow net (in < 10 min) for the ground water flow under the sheet pile wall shown on the attached page.

1) Identify equipotential lines and streamlines before starting.

2) Draw your flow net – stop after < 10 min.

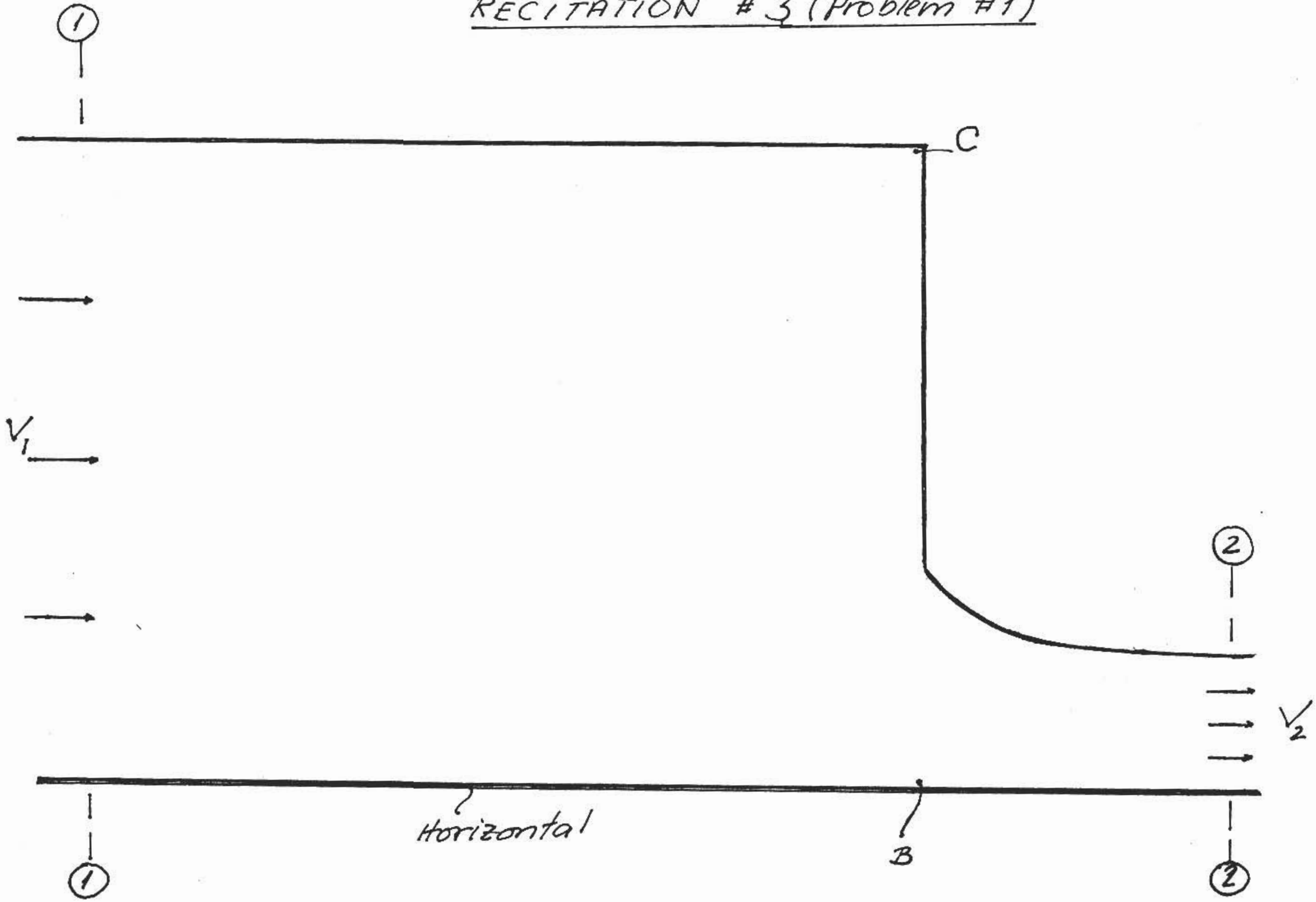
3) Determine the number of stream tubes, N_f, obtained from your flow net.

4) Determine the number of equipotential head drops, N_H, from your flow net.

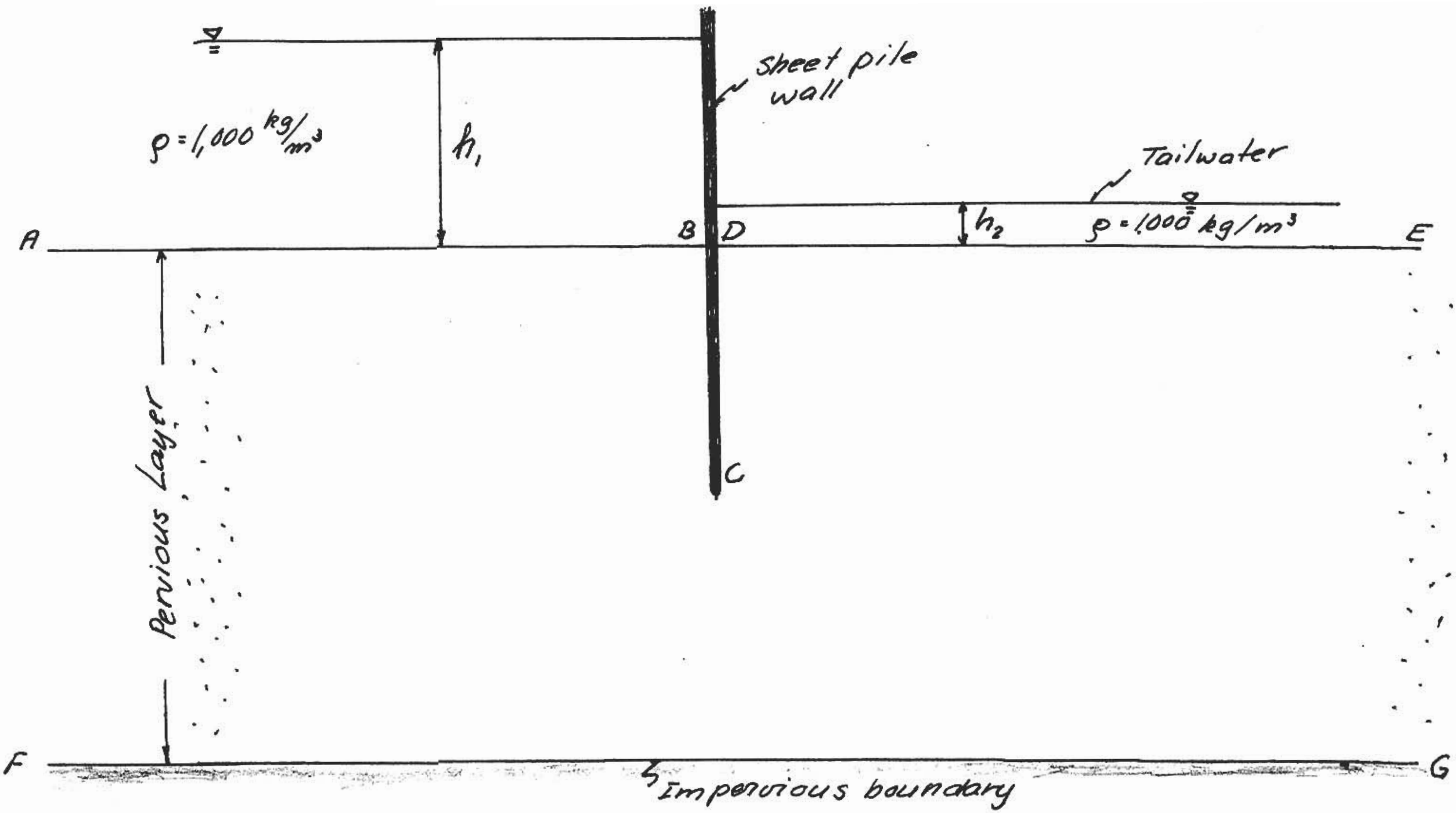
$$Q = K(h_1 - h_2)\frac{N_f}{N_H}$$

5) What is your 10 min answer for the important quantity N_f / N_H?

RECITATION #3 (Problem #1)



Problem #2



$\rho = 1{,}000\ kg/m^3$
h_1
Sheet pile wall
Tailwater
h_2
$\rho = 1000\ kg/m^3$
A
B
D
E
C
Pervious Layer
F
G
Impervious boundary

RECITATION #3

Problem #1

a)

For a "perfect" flow net see attached. Note that your flow net was for a conduit, whereas the attached is for a free surface flow under a gate [main difference is that the upper boundary is a free surface, which is a stream line along which $p=0$ and
z varies, and not a horizontal wall along which $z=\text{const.}$ and pressure may vary].

b)

Well behaved flow at 1-1 and 2-2 means that volume conservation yields

$$Q_{in} = V_1 h_1 = Q_{out} = V_2 h_2 \Rightarrow \underline{V_2/V_1} = \frac{h_1}{h_2} = \frac{3.5}{0.75} = \underline{4.67}$$

c)

If $p=0$ at 1-1 and 2-2 on upper boundary [its a free surface in reality, so $p=0$ is correct] and flow is well behaved with straight parallel streamlines, the pressure varies hydrostatically along lines ⊥ streamlines - in this case this direction is z. So, we have

$$p_1 + \rho g z_1 = \rho g h_1 \quad \text{anywhere along 1-1}$$

and

$$p_2 + \rho g z_2 = \rho g h_2 \quad \text{anywhere along 2-2}$$

Bernoulli along any streamline from 1-1 to 2-2 therefore gives

$$p_1 + \rho g z_1 + \frac{1}{2}\rho V_1^2 = p_2 + \rho g z_2 + \frac{1}{2}\rho V_2^2$$

or

$$V_2^2 - V_1^2 = 2g\,(h_1 - h_2) \qquad (1)$$

but

$$V_2 = Q/h_2 \text{ and } V_1 = Q/h_1 \quad [\text{from (b)}]$$

So

$$Q^2\left[\frac{1}{h_2^2} - \frac{1}{h_1^2}\right] = 2g(h_1 - h_2) \Rightarrow \underline{Q = 5.64\,\frac{m^3}{s}\text{ per m.}}$$

$$\underline{V_1 = \frac{Q}{h_1} = 1.61\,m/s} \;;\; \underline{V_2} = \frac{Q}{h_2} = V_1 \cdot \frac{h_1}{h_2} = \underline{V_1 \cdot 4.67 = 7.52\,m/s}$$

If we neglect V_1^2 in Eq. (1) – after all we got $V_1 = V_2/4.67$ in (b) – we have

$$V_2^2 \cong 2g\,(h_1 - h_2) \Rightarrow \underline{V_2} \cong \sqrt{2g(h_1 - h_2)} = \underline{7.34\,m/s}$$

(not a bad approximation) $V_1 = V_2/4.67 = \underline{1.57\,m/s}$

d)

Along top streamline $p + \rho g z + \frac{1}{2}\rho V^2 =$
$p_1 + \rho g z_1 + \frac{1}{2}\rho V_1^2 = 0 + \rho g z_1 + \frac{1}{2}\rho V_1^2 = p_c + \rho g z_c + \frac{1}{2}\rho V_c^2.$

Now, $z_1 = z_c$ and $V_c = 0$ since "corner" at z points away from flow.

Thus,

$$p_c = \frac{1}{2}\rho V_1^2 = \frac{1}{2}\,1{,}000\;1.61^2 = 1.3\cdot 10^3\,Pa$$

or

$$\underline{p_c/\rho g} = \text{pressure head at C} = \underline{0.13\,m}$$

[Notice: When upper boundary is a free surface this pressure at elevation $z_1 = z_c$ is achieved by an increase in the free surface elevation of 0.13m at C.]

e)

Along the bottom, which is a streamline, we have from Bernoulli from 1-1 to a point, BG, on the bottom immediately below the vertical gate

$$p_1 + \rho g z_1 + \frac{1}{2}\rho V_1^2 = p_{BG} + \rho g z_{BG} + \frac{1}{2}\rho V_{BG}^2$$

$$p_1 = \rho g h_1 \quad ; \quad z_1 = z_{BG} = 0$$

$$p_{BG} = p_1 - \frac{1}{2}\rho\left(V_{BG}^2 - V_1^2\right) = \rho g h_1 - \frac{1}{2}\rho\left(V_{BG}^2 - V_1^2\right)$$

Measurement directly on flow net gives

$$2\Delta S_{BG} = 18\,mm \Rightarrow \Delta S_{BG} = \Delta h_{BG} = 9\,mm$$

$$\Delta S_{1-1} = \Delta h_{1-1} = 30\,mm$$

so volume conservation in bottom streamtube gives

$$V_{BG} \cdot \Delta h_{BG} = V_1 \Delta h_{1-1} \Rightarrow V_{BG} = \frac{\Delta h_{1-1}}{\Delta h_{BG}} V_1 = \frac{30}{9} V_1$$

$V_1 = 1.61\ m/s$ from (c), so $V_{BG} = 5.37\ m/s$

and

$$\underline{p_{BG}} = \rho g h_1 - \frac{1}{2}\rho\left(\left(\frac{30}{9}\right)^2 - 1\right)V_1^2 = (34.3 - 13.1)\,kPa = \underline{21.2\,kPa}$$

Thus, pressure head on bottom below gate =

$$\underline{\frac{p_{BG}}{\rho g} = 2.16\ m}$$

which is larger (by nearly a factor of 2) than corresponding to the depth of flow under gate $h_G \approx 1.2\,m$.

FREE OUTFLOW UNDER A GATE

Q = discharge per unit length into paper: UNKNOWN
ΔQ = discharge in stream tubes = $Q/4$
V_1 = upstream velocity = Q/h_1 (depth h_1 known)
V_2 = downstream velocity = Q/h_2 (depth h_2 known)
$p + \rho g z + \frac{1}{2}\rho V^2 = \rho g h_1 + \frac{1}{2}\rho V_1^2$ for all streamlines
when $z = 0$ is chosen along horizontal bottom
$V = \Delta Q/\Delta h$ can be estimated from flow net
With V and z specified, Bernoulli gives p.

$$p = \rho g(h_1 - z) + \frac{1}{2}\rho(V_1^2 - V^2)$$

At downstream outflow section flow is well behaved

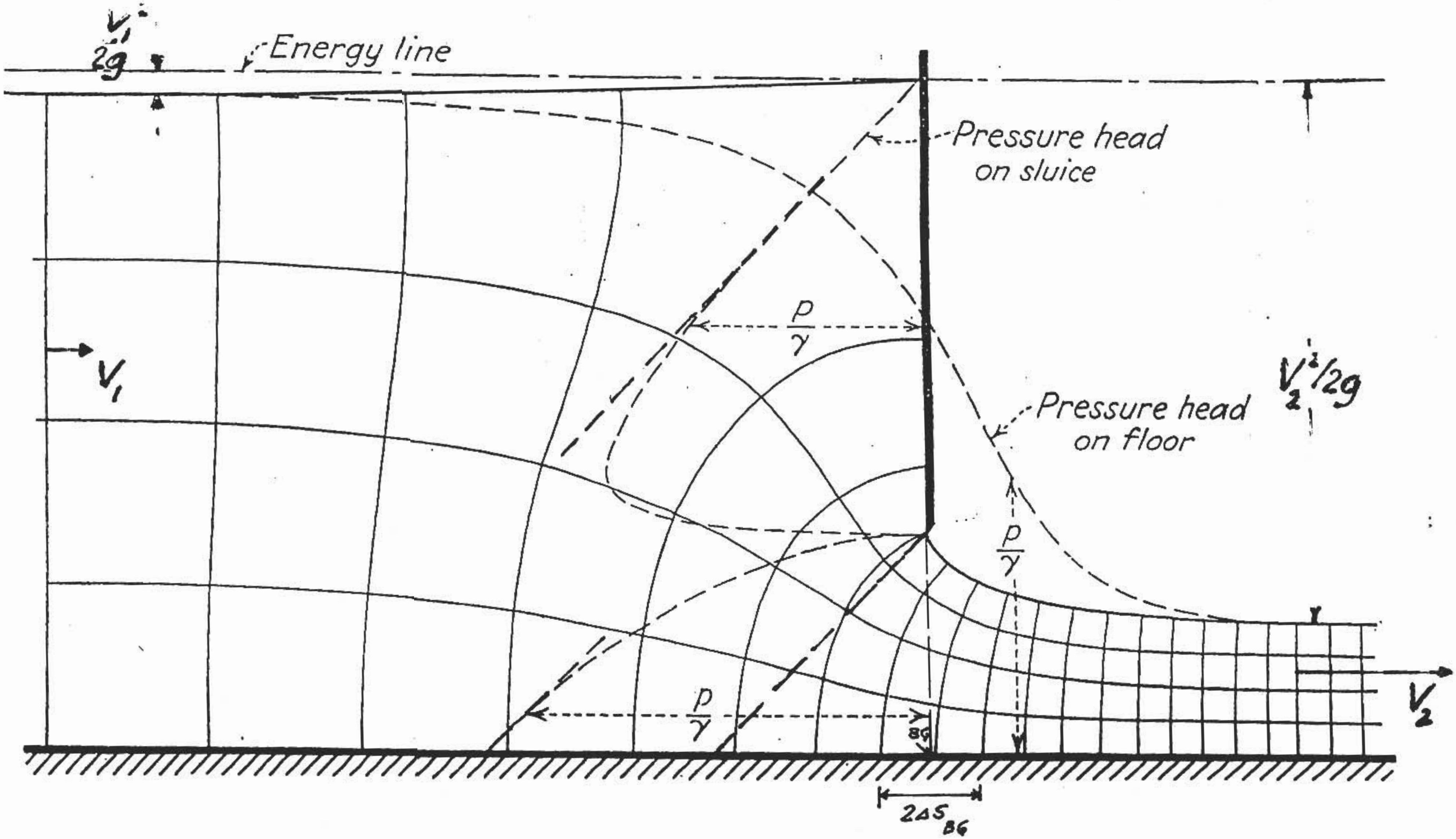
$$p_2 + \rho g z_2 = h_2$$

and therefore

$$V_2^2 - V_1^2 = Q^2\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right) = 2g(h_1 - h_2) \Rightarrow Q\,!$$

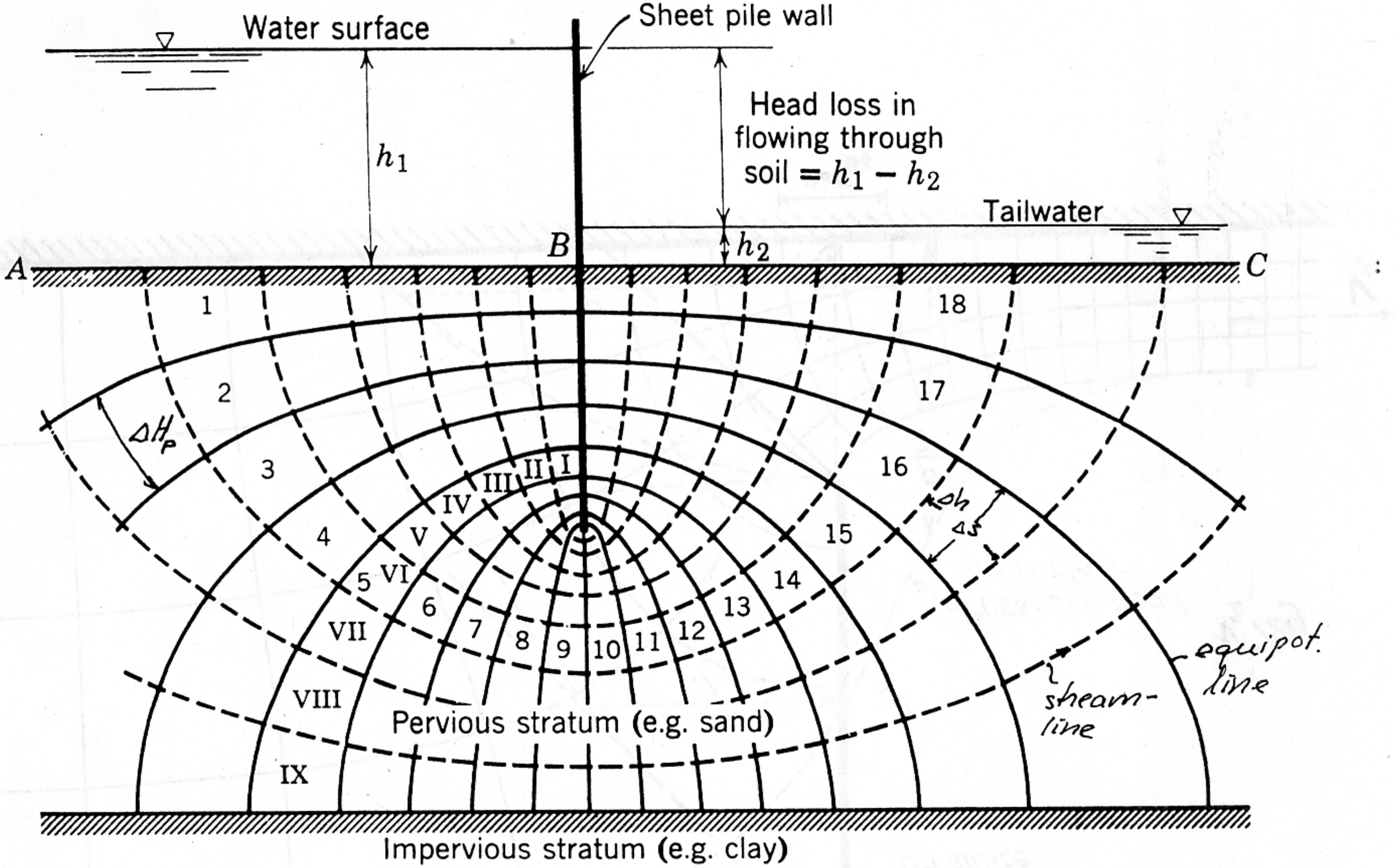
Note Flow Net Features:

1) Increase in free surface elevation at gate ($V = 0$ in corner)
2) Free surface, $p = 0$, is same as Piezometric Head, z_p
3) Pressure on gate ~hydrostatic near top (since $V \approx 0$) but drops below closer to opening (since $V \neq 0$)
4) Pressure directly under gate increases faster than hydrostatic near surface because streamlines are curved (pressure force needed to accelerate flow)
5) Outflow depth, h_2, is smaller than gate opening because flow can not turn a sharp corner.
6) $H = V^2/2g + p/\rho g + z$ = CONSTANT is achieved by large h_1 and small V_1 becoming small h_2 and large V_2.



$\frac{V_1^2}{2g}$
Energy line
Pressure head on sluice
$\frac{p}{\gamma}$
V_1
$V_2^2/2g$
Pressure head on floor
$\frac{p}{\gamma}$
$\frac{p}{\gamma}$
BG
V_2
$2\Delta s_{BG}$

Seepage Flow Net: Problem No:2



N_H = number of headdrops (18); $\Delta H = (h_1 - h_2)/N_H$

N_f = number of flow channels (9); K = Hydraulic Conductivity [m/s]

Q = dischange under wall = $K(h_1 - h_2) N_f / N_H$ (m^3/s per m)