1.061 / 1.61 Transport Processes in the Environment
Fall 2008

Solution 1

A convenient control volume can be drawn around the interior volume of the tank, and extending into pipes 1 and 2 to positions of uniform concentration, i.e. $\frac{\partial C}{\partial n} = 0$ along pipe.

DASHED LINE DEFINES
CONTROL VOLUME
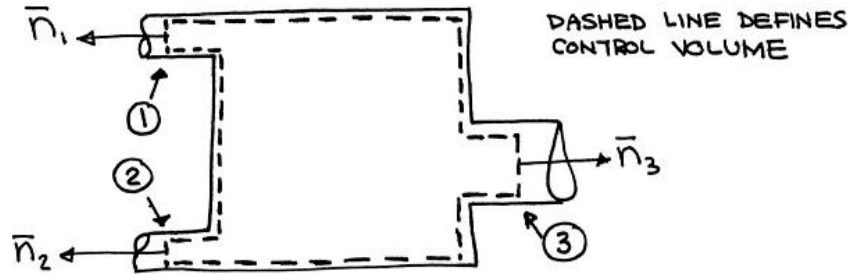
Now evaluate Eq.4 for this control volume.

(A)

$$\frac{\partial}{\partial t} \int_{C\forall} C d\forall = -\int_{CS} C\vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

Because we assume steady state, $\frac{\partial}{\partial t} = 0$, the first term is zero. No source or sink is mentioned, $\therefore$ set $S = 0$. We evaluate the two surface integrals, $\int_{CS}$, at the three indicated areas of flux. Note that we placed the surface $1, 2, 3$ far enough into the pipes that $\frac{\partial C}{\partial n} = 0$ at each surface. $\therefore$ there is no diffusive flux, $\int_{CS} D_n \frac{\partial C}{\partial n} dA = 0$.

(B) Evaluating $\int_{CS} C\vec{V} \cdot \vec{n} dA$ at each flux area,

$$0 = +u_1 A_1 C_1 + u_2 A_2 C_2 - u_3 A_3 C_3.$$

From conservation of fluid mass (continuity), we also have $u_1 A_1 + u_2 A_2 = u_3 A_3$ for incompressible flow.

(C) Using this to replace $u_3 A_3$ in (B) and solving for C_3,

$$C_3 = \frac{u_1 A_1 C_1 + u_2 A_2 C_2}{(u_1 A_1 + u_2 A_2)}$$

or,

$$C_3 = \frac{(20\,cm/s)(10\,cm^2)(9\,mg/l) + (10\,cm/s)(10\,cm^2)(0\,mg/l)}{(20\,cm/s)(10\,cm^2) + (10\,cm/s)(10\,cm^2)}$$

$$C_3 = \frac{20}{30} * 9\,mg/l = 6\,mg/l$$

Solution 2

Apply the integral form of mass conservation to the control volume indicated by dashes.

(A)
$$\frac{\partial}{\partial t} \int_{C\forall} C d\forall \qquad = - \int_{CS} C\vec{V} \cdot \vec{n} dA \qquad + \int_{CS} D_n \frac{\partial C}{\partial n} dA \qquad + S$$

Ø, b/c we assume   evaluate at surface   Ø, b/c we place   given

steady state   sections 1,2   surfaces   1   &   2

                                           where $\frac{\partial C}{\partial n} = 0$
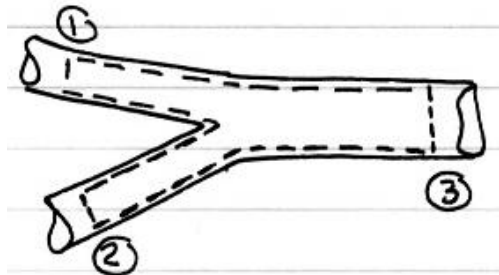
(B)
$$0 = u_1 C_1 A_1 - u_2 C_2 A_2 + S$$

Note: From continuity, $u_2 = u_1$, because $A_2 = A_1$

(C)
$$C_2 = \frac{S}{u_2 A_2} = \frac{S}{u_1 A_1} = \frac{5\,g/s}{(10\,cm/s)(10\,cm^2)} = 50\,mg/cm^3$$

Solution 3

Choose a control volume (dash) far enough away from juncture such that $\frac{\partial T}{\partial n} = 0$ at each flux surface.

(A)
$$\frac{\partial}{\partial t} \int_{CV} C d\forall = - \int_{CS} C\vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA \pm S$$

The concentration of heat energy is,

$$C[Jm^{-3}] = \rho c_p T$$

fluid density   specific heat   temp

$[kgm^{-3}]$ $[Jkg^{-1}K^{-1}]$ $[K]$

$$C_p = 4200\,Jkg^{-1}K^{-1} \text{ for water}$$

For simplicity, assume $\rho, c_p \neq f(T)$. If we assume steady state, then the first term in (A) is zero. Because we position surfaces 1, 2, 3 where $\frac{\partial T}{\partial n} = 0$, the diffusive flux term is zero. Because the pipes are insulated, $S = 0$. So, finally (A) becomes,

(B)
$$0 = \rho c_p T_1 u_1 A_1 + \rho c_p T_2 u_2 A_2 - \rho c_p T_3 u_3 A_3$$

Dropping ρc_p, and solving for T_3,

(C)
$$T_3 = \frac{u_1 A_1 T_1 + u_2 A_2 T_2}{u_3 A_3}$$

Note from statement $u_1 A_1 = u_2 A_2$. And from fluid mass conservation $(u_1 A_1 + u_2 A_2) = u_3 A_3$.

(D)
$$T_3 = \frac{u_1 A_1 (T_1 + T_2)}{2 u_1 A_1} = \frac{1}{2}(T_1 + T_2)$$

$\therefore T_3 = 15 \deg C \quad [288K]$

Solution 4
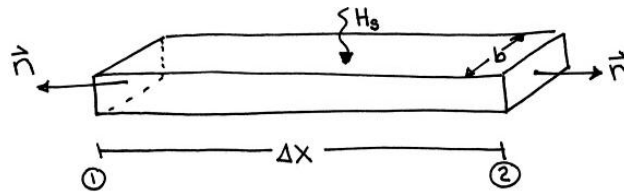
Control Volume Approach:

Select a short length of river, ΔX, and evaluate the control volume (integral) form of the conservation equation. For conservation of heat energy, replace $C = \rho c_p T$ in Eq.4.

(A)
$$\frac{\partial}{\partial t} \int_{CV} \rho c_p T \, d\forall = -\int_{CS} \rho c_p T \vec{V} \cdot \vec{n} dA + \int_{CS} D_n \frac{\partial C}{\partial n} dA + H_s \Delta x b$$

As the problem statement does not indicate any unsteadiness, we assume steady flow, i.e. $\frac{\partial}{\partial t} = 0$.

(B) Evaluating the flux terms in (A),

$$0 = -\rho c_p U b h (T_2 - T_1) + \rho c_p D b h \left(\left.\frac{\partial T}{\partial X}\right|_2 - \left.\frac{\partial T}{\partial X}\right|_1 \right) + H_3 \Delta x b$$

3

Using a Taylor expansion, assuming T is continuous in X,

$$T_2 = T_1 + \frac{\partial T}{\partial X}\Delta X$$

$$\left.\frac{\partial T}{\partial X}\right|_2 = \left.\frac{\partial T}{\partial X}\right|_1 + \frac{\partial}{\partial X}\left(\frac{\partial T}{\partial X}\right)\Delta X$$

Plug these expansions in (B), and divide out the common terms, Δxb.

(C)
$$0 = \rho c_p h\left(-U\frac{\partial T}{\partial X} + D\frac{\partial^2 T}{\partial X^2}\right) + H_s$$

From which, one could solve for $\frac{\partial T}{\partial X}$.

It is useful to consider the relative importance of the advective and diffusive fluxes. Here, specifically the relative magnitudes of $U\frac{\partial T}{\partial X}$ and $D\frac{\partial^2 T}{\partial X^2}$. The scale of each term can be estimated from this system. Consider the control volume length, Δx, as an appropriate length-scale, then

$$U\frac{\partial T}{\partial X} \sim U\frac{\Delta T}{\Delta X}$$

$$D\frac{\partial^2 T}{\partial X^2} \sim D\frac{\Delta T}{\Delta X^2}$$

Where ΔT is the temperature change across ΔX. The relative magnitude of these terms is then,

$$\frac{\text{advective flux}}{\text{diffusive flux}} = \frac{U\frac{\Delta T}{\Delta X}}{D\frac{\Delta T}{\Delta X^2}} = \frac{U\Delta X}{X}$$

This dimensionless parameter is called the peclet number. It is discussed in detail in Chapter 5.

If $\frac{U\Delta X}{D} >> 1$,   then advective fluxes dominate diffusive fluxes, and we can drop the term $D\frac{\partial^2 T}{\partial X^2} << U\frac{\partial T}{\partial X}$.

If $\frac{U\Delta X}{D} << 1$,   diffusive fluxes $\left(D\frac{\partial^2 T}{\partial X^2}\right)$ are much larger than advective fluxes $\left(U\frac{\partial T}{\partial X}\right)$, and we can drop $U\frac{\partial T}{\partial X}$.

Since ΔX is not specifically defined, we ask, e.g., for what length-scale will advection dominate transport?

$$\frac{U\Delta X}{D} >> 1 \text{ iff } \Delta X >> \frac{D}{U} = \frac{0.1\,m^2s^{-1}}{1\,m^2s^{-1}} = 0.1\,m$$

4

$\therefore$ over any length-scale, $\Delta X \gg 0.1\,m$, we may neglect the impact of diffusion in (A) for this system. The problem asks for a description of $\dfrac{\partial T}{\partial X}$ along a river channel. In such a system, the length scales of interest are much larger than 10 cm, and are more like 100 m to km's. Therefore, for this system, we can safely drop the diffusive transport term. Then, (C) reduces to,

(D)
$$0 = -\rho c_p h U \frac{\partial T}{\partial X} + H_s$$

from which,

(E)
$$\frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h U} = \frac{J s^{-1} m^{-2}}{(kg m^{-3})(J kg^{-1} K^{-1})(m)(m s^{-1})} = \frac{K}{m}$$

Using the stated parameters,

(F)
$$\frac{\partial T}{\partial X} = \frac{800\,W m^{-2}}{(1000\,kg/m^3)(4200\,J/kgK)(1\,m)(1\,m/s)} = 2 \times 10^{-4} K/m = 0.2\,C/km$$

Differential Approach:

The conservation of mass equation can be applied to the transport of heat energy by noting the concentration of heat energy, $C[J/m^3] = \rho c_p T$. Then, the differential form of the conservation equation is, for incompressible flow,

(G)
$$\frac{\partial}{\partial t}(\rho c_p T) + U\frac{\partial}{\partial x}(\rho c_p T) + V\frac{\partial}{\partial y}(\rho c_p T) + W\frac{\partial}{\partial z}(\rho c_p T)$$
$$= \frac{\partial}{\partial x}D_x\frac{\partial}{\partial x}(\rho c_p T) + \frac{\partial}{\partial y}D_y\frac{\partial}{\partial y}(\rho c_p T) + \frac{\partial}{\partial z}D_z\frac{\partial}{\partial z}(\rho c_p T) \pm S$$

- If we neglect $\rho = f(T)$, then $\rho \neq f(x, y, z, t)$.

- The problem statement gives us $V = W = 0$, and isotropic, homogeneous $D = D_x = D_y = D_z \neq f(x, y, z)$.

- If we assume the system is uniform (well-mixed) in y and z, then $T \neq f(y, z)$.

- The source term is given as a surface flux, $H_s = [J s^{-1} m^{-2}]$. Since the equation deals in volume concentration, we must divide by depth to put the source term in consistent units.
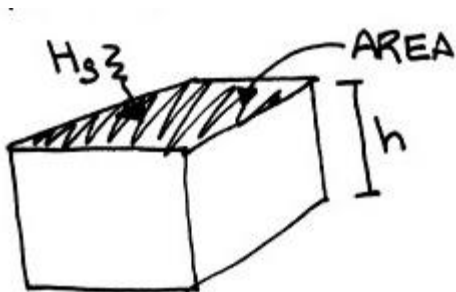
$$H_s = \frac{J/m^2}{s} = \frac{\text{energy per surface area}}{s}$$

$$\frac{H_s}{h} = \frac{J/m^3}{s} = \frac{\text{energy per volume}}{s}$$

Applying the above points, (G) reduces to

(H)

$$\rho c_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial X} \right] = D \frac{\partial^2 T}{\partial X^2} + \frac{H_s}{h}$$

For typical length scales of interest along a river channel, $\Delta X \sim 100m$ to km's, it is easy to show that the diffusion term, $D\frac{\partial^2 T}{\partial X^2}$, is small compared to $U\frac{\partial T}{\partial X}$, the advection term. Thus we will drop $D\frac{\partial^2 T}{\partial X^2} << U\frac{\partial T}{\partial X}$.
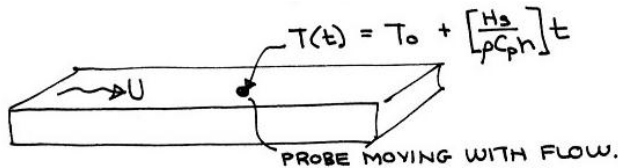
See scaling arguments given above.

Finally, note that the bracketed term in (H) is the total derivative.

(I)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} = \frac{H_s}{\rho c_p h} = \left[\frac{°C}{T} \right]$$

This equation may be read in the Lagrangian context as, following a particular fluid particle, we would observed its temperature to increase at the rate $\left[\frac{H_s}{\rho c_p h} \right]° C/s$.

$$T(t) = T_o + \left[\frac{H_s}{\rho c_p h} \right] t$$
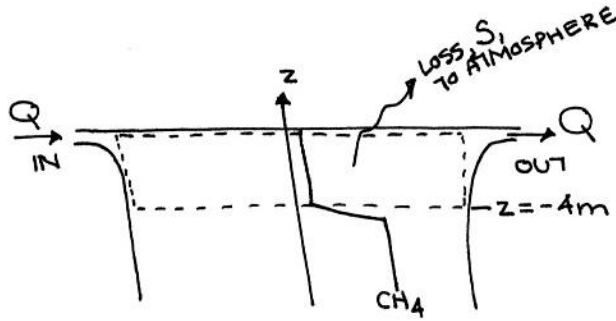
PROBE MOVING WITH FLOW.

If the flow/thermal conditions are steady, $\frac{\partial T}{\partial t} = 0$, then (I) also provides a simple description of spatial gradient.

(J)

$$\frac{\partial T}{\partial X} = \frac{H_s}{U \rho c_p h} = \left[\frac{°C}{L} \right]$$

Solution 5

Define the epilimnion as the control volume of interest. The conservation of mass for this system, assuming steady state, is

(A)

$$0 = (QC)_{IN} - (QC)_{OUT} - D_Z \frac{\partial C}{\partial Z} A - S$$

given $C_{IN} = 0$. Assume epilimnion is well-mixed. $\therefore C_{OUT} = C_{EPI} = C(z = -4\,m)$. Discretize concentration gradient across thermocline,

$$\left. \frac{\partial C}{\partial Z} \right|_{z=-4} \approx \frac{[C(z=-4) - C(z=-5)]}{[-4 - -5]}$$

Then solve (A) for S,

(B)

$$S = -QC_{EPI} - D_Z A \frac{C_{EPI} - C_{HYPO}}{1\,m}$$

$$= - \left(1\,m^3/s\right)\left(0.4\,mg/l\right) - \left(2 \times 10^{-6}\,m^2 s^{-1}\right)\left(10^6\,m^2\right)\left[\frac{0.4 - 4\,mg/l}{1\,m}\right]$$

$$= 6.8 \frac{m^3}{s} \frac{mg}{l} \frac{1000\,l}{m^3} = 6.8\,g/s$$
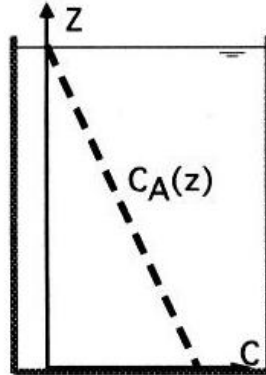
Solution 6

From Fick's Law, $q_z = -DA\frac{\partial C}{\partial Z}$.

(1) At z = 0 and z = H, $\frac{\partial C}{\partial Z} < 0$, $\therefore q_z > 0$, which indicates fluxes is upward at both boundaries.

(2) At steady state, $\frac{\partial C}{\partial t} = 0$. From 1-D conservation equation,

$$\frac{\partial C}{\partial t} = D_z \frac{\partial^2 C}{\partial Z^2}$$

$\therefore$ at steady state, $\dfrac{\partial^2 C}{\partial Z^2} = 0.$ $\therefore$ $\dfrac{\partial C}{\partial Z} =$ constant. $\therefore$ linear profile indicates this system is at steady state.
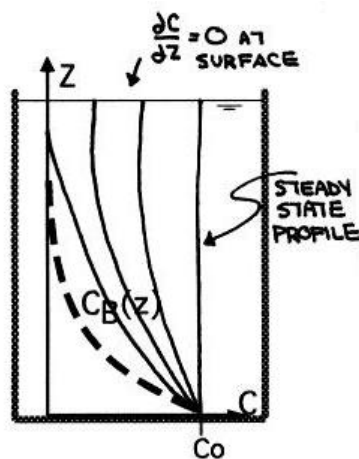

$C_A(z)$

(1) $\dfrac{\partial C}{\partial Z} < 0$ at z = 0. $\therefore q_z > 0$, positive, upward. No flux at z = H.

(2) System is not at steady state.

$$\frac{\partial}{\partial Z}\left(\frac{\partial C}{\partial Z}\right) \neq 0, \therefore \text{ from conservation EQ, } \frac{\partial C}{\partial Z} \neq 0$$

Progression of evolving profiles shown. Note, because chemical is not volatile, surface (z = H) is a no flux boundary. $\therefore$ $\left.\dfrac{\partial C}{\partial Z}\right|_{z=H} = 0$ so that $q_z(z = H) = 0!$

$\frac{\partial C}{\partial Z} = 0$ AT SURFACE

STEADY STATE PROFILE

$C_B(z)$

C_o

8

Solution 7

(a) The general equation (14b) is

$$\underbrace{\frac{\partial C}{\partial t}}_{(1)} + \underbrace{u\frac{\partial C}{\partial x}}_{(2)} + \underbrace{v\frac{\partial C}{\partial y}}_{(3)} + \underbrace{w\frac{\partial C}{\partial z}}_{(4)} = D[\underbrace{\frac{\partial^2 C}{\partial x^2}}_{(5)} + \underbrace{\frac{\partial^2 C}{\partial y^2}}_{(6)} + \underbrace{\frac{\partial^2 C}{\partial z^2}}_{(7)}] \underbrace{\pm S}_{(8)}$$

We can eliminate several terms in this equation:

(1) because we are looking at steady-state conditions,

(2), (3), (5) and (6) due to the negligible horizontal gradients of DO concentration and

(4) due to the insignificant vertical flows.

Thus, the equation reduces to

$$0 = D\left[\frac{\partial^2 C}{\partial z^2}\right] - S$$

noting that phytoplankton are a sink of dissolved oxygen. For first-order removal, $S = kC$, so our governing equation is:

$$D\left[\frac{\partial^2 C}{\partial z^2}\right] - kC = 0$$

(b) The general solution to this governing equation is:

$$C(z) = Aexp\left(z\sqrt{\frac{k}{D}}\right) + Bexp\left(-z\sqrt{\frac{k}{D}}\right)$$

where A and B are constants. We use our boundary conditions to determine A and B.

- $C|_{z=0} = 8 \Rightarrow A + B = 8$

- No flux at $z = -10\,m \Rightarrow \left.\frac{\partial C}{\partial z}\right|_{z=-10} = 0.$

Therefore,

$$A\sqrt{\frac{k}{D}}exp\left((-10\,m)\sqrt{\frac{k}{D}}\right) - B\sqrt{\frac{k}{D}}exp\left(-(-10\,m)\sqrt{\frac{k}{D}}\right) = 0$$

$$\Rightarrow Aexp\left((-10\,m)\sqrt{\frac{k}{D}}\right) = Bexp\left((10\,m)\sqrt{\frac{k}{D}}\right).$$

Using our values for k and D,

$$\sqrt{\frac{k}{D}} = \sqrt{\frac{0.05\,day^{-1}}{0.1\,cm^2s^{-1}}} = \sqrt{\frac{5.787 \times 10^{-7}\,s^{-1}}{10^{-5}\,m^2s^{-1}}} = 0.240/m$$

Therefore, from our second boundary condition,

$$Aexp\left(-2.4\right) = Bexp(2.4) \Rightarrow 0.091A = 11.0B \Rightarrow A = 121.5B$$

Solving the boundary condition equations simultaneously gives A = 7.935, B = 0.065.

Therefore,

$$C(z) = 7.935exp(0.240z) + 0.065exp(-0.240z) \quad \text{[for z in meters]}$$

(c) As we have only a DO sink in the water column, the lowest concentration of DO will be found at the point furthest from the surface, which is maintained at a constant concentration. Thus, we expect the DO concentration to be lowest at z = -10 m. This is confirmed by looking at the plot of the vertical profile of DO in the water column (note that we have satisfied both boundary conditions).
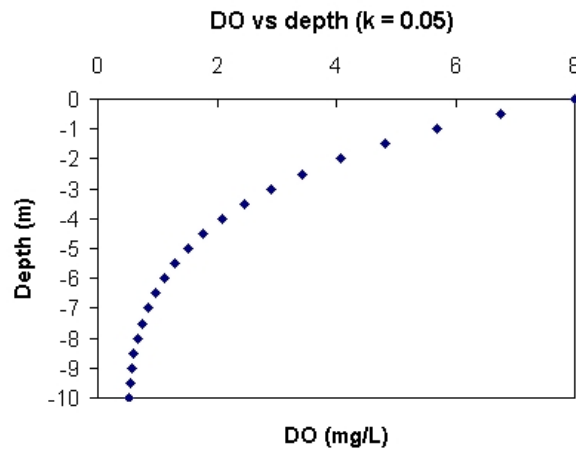
(d) As we can see from the above plot, $C_{min} = 1.4\,mg/L$. However, if $k = 0.1\,day^{-1}$, repetition of the above calculation yields a minimum concentration of approximately $0.5\,mg/L$, meaning the the health of Lake Monger is under threat.