Logistical and Transportation Planning Methods

1.203J/6.281J/13.665J/15.073J/16.76J/ESD.216J

Massachusetts Institute of Technology
Cambridge, Massachusetts

Quiz #1
October 26, 2005

OPEN BOOK
TWO HOURS
5 PAGES, 3 PROBLEMS

PLEASE SHOW ALL YOUR WORK!

Inspirational quote:

You've got to be very careful if you don't know where you are going because you might not get there.

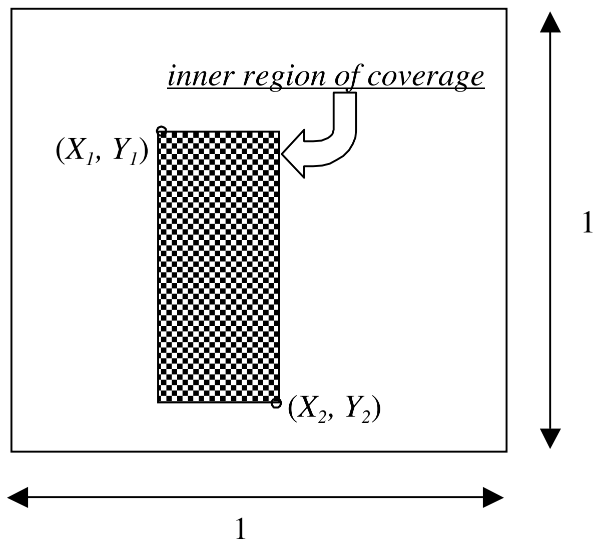
Yogi Berra, once a baseball catcher for the New York Yankees

35 points. Problem 1. Regions of Inner Coverage

Consider two random points, (X_1, Y_1) and (X_2, Y_2), located uniformly and independently within a unit area square. These points are the locations of two independently patrolling police cars. Travel distance D_{12} between the two points is the usual Manhattan or right-angle or L_1 distance metric,

$$D_{12} = | X_1 - X_2 | + | Y_1 - Y_2 |$$

The 'inner region of coverage' formed by these two police cars at any time is the set of points in the rectangle formed by their locations, as shown in the figure.



We are interested in the area of the *inner region of coverage*.

(a) **10 pts.** Find the mean area of the *inner region of coverage*.
(b) **10 pts.** Find the variance of the area of the *inner region of coverage*.
(c) **15 pts.** We are interested in the pdf of the area of the *inner region of coverage*. In our four-step process to obtain this, carefully work through steps 1, 2 and 3, and set up carefully and precisely but (in Step 4) do not compute any integrals you develop.

35 points. Problem 2: M/M/1 with a Variation

Consider a queueing system that is very similar to a M/M/1 system with infinite queue capacity but has one special feature. The system has Poisson arrivals with a rate of λ per unit of time. Service times, however, differ depending on whether an arriving customer finds the server busy or not.

Specifically, if an arriving customer finds the server idle (and the system empty), this customer's service time is negative exponential with an expected value equal to $1/\mu_I$. (Note that in this case, the newly-arrived customer is immediately admitted into service.)

If, however, an arriving customer finds the server busy (and, thus, has to join a queue) his/her service time is negative exponential with an expected value equal to $1/\mu$ (i.e., different from $1/\mu_I$).

[Please note again that only the customers who find the server idle have an expected service time of $1/\mu_I$.]

(a) **8 pts.** Suppose the server is currently idle. What is the probability that more than one customers will be served during the next busy period of the server (i.e., before the server becomes idle again)?

(b) **8 pts.** Suppose the server is currently idle. What is the probability that the next busy period will consist of exactly three services to customers (followed by another idle period)?

(c) **8 pts.** Is the condition $\lambda < \mu$ sufficient to guarantee that this system will reach steady-state (equilibrium)? Please justify your answer in one or two sentences.

(d) **11 pts.** Please draw carefully a state-transition diagram for this queueing system, that would permit determining all the steady-state probabilities (assuming steady-state is reached). The key here is to define carefully the states of the system. Please make sure to do so on your paper. DO NOT write any balance equations or try to solve for the probabilities. All we are looking for is the state-transition diagram.

30 points. Problem 3. Hurricane Xenos

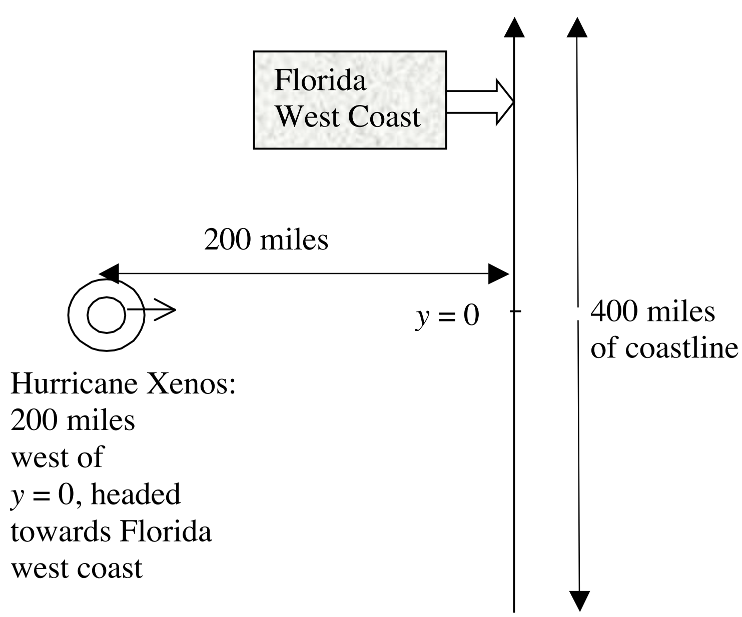
The west coastline of Florida is modeled as a finite length straight line. Let us call it the 'y-axis' and we consider it to run from South to North, with increasing values of y associated with moving North. See the figure. We are concerned with a hurricane that will hit this west coast.

For simplicity we model the eye or center of a hurricane as a single point. At any given time meteorologists have a conditional pdf for the location on the y-axis at which the eye of the next hurricane, "Hurricane Xenos" will pass over. This is a Cauchy pdf, as derived in class and in the book, with the peak of the Cauchy pdf at $y = 0$. Since we are dealing with a finite length coastline, the Cauchy must be modified in two ways. First, the Cauchy we are dealing here is a conditional pdf to account for the fact that y is restricted to values between +200 and –200 miles. Second, the point equivalent to 'the flashlight spinning' is not unit distance from the axis, but rather 200 miles, as shown in the figure. **FOR EASE OF OPEN BOOK USE, WE HAVE PLACED THE BOOK DERIVATION OF THE CAUCHY ON THE NEXT PAGE.**

(a) **10 pts.** Write an expression for the conditional pdf of the location on the Florida coastline that the eye of the hurricane will hit, given that it will hit the Florida coastline.

(b) **10 pts.** Is the mean of the conditional pdf found in (a) well defined? Why or why not? Is the variance finite? Why or why not?

(c) **10 pts.** Strong hurricane-force winds extend for miles in both directions from the eye. So citizens and buildings at a point y may be in trouble even if the eye does not pass over point y directly. The *hazard score* for any point y is the probability that that the eye of the hurricane passes within 30 miles to the south of y or within 40 miles to the north of y, given that the hurricane will hit the Florida coast. Find an expression for the hazard score for the point y =100.



BOOK DERIVATION OF THE CAUCHY CDF AND PDF.

$$F_X(x) \equiv P\{X \leq x\} = P\{\tan \Theta \leq x\}$$
$$= P\{\Theta \leq \tan^{-1} x\}$$

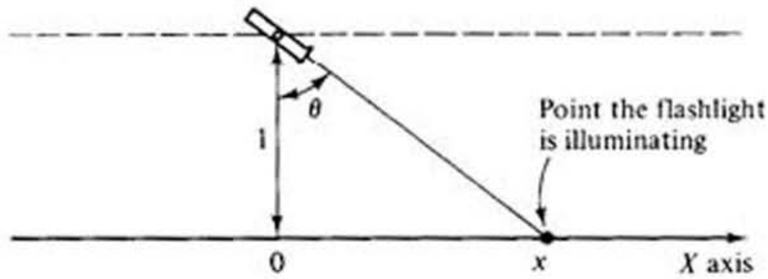


FIGURE 3.23 Spinning the flashlight.

Since Θ is uniformly distributed over $[-\pi/2, \pi/2]$,

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{\pi} & -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = P\{\Theta \leq \tan^{-1} x\} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x$$

The pdf is

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \qquad -\infty < x < +\infty \qquad (3.57)$$