Spatially Distributed Queues II

M/G/1

2 Servers

N servers: Hypercube Queueing Model

Approximations

Setup: Hypercube Queueing Model

- Region comprised of geographical atoms or nodes
- Each node *j* is an independent Poisson generator, with rate λ_j
- Travel times: τ_{il} = travel time from node *i* to node *j*
- *N* servers
- Server locations are random: l_{nj}

Setup: Hypercube Queueing Model - con't.

- Server assignment: one assigned
- State dependent dispatching
- Service times: mean = $1/\mu_n$; *negative exponential density*
- Service time dependence on travel time
- We allow a queue (FCFS, infinite capacity)

Fixed Preference Dispatch Policies for the Model

- Idea: for each atom, say Atom 12, there exists a vector of length *N* that is the preference-ordered list of servers to assign to a customer from that atom
- Example: {3,1,7,5,6,4,2}, for *N*=7.
- Dispatcher always will assign the most preferred ***available*** server to the customer
- Usually order this list in terms of some travel time criterion.

Example Dispatch Policies

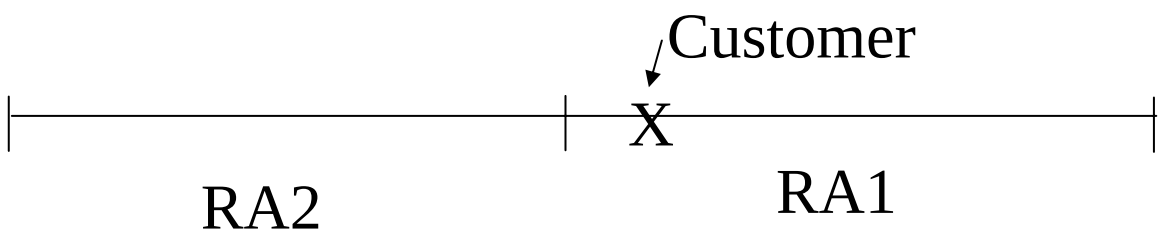
- SCM: Strict Center of Mass
  - Place server at its center of mass
  - Place customer at its center of mass
  - Estimate travel times: center of mass to center of mass
- MCM: Modified Center of Mass
  - Place server at its center of mass
  - Keep customer at centroid of atom
  - Estimate travel times: center of mass to centroid of atom

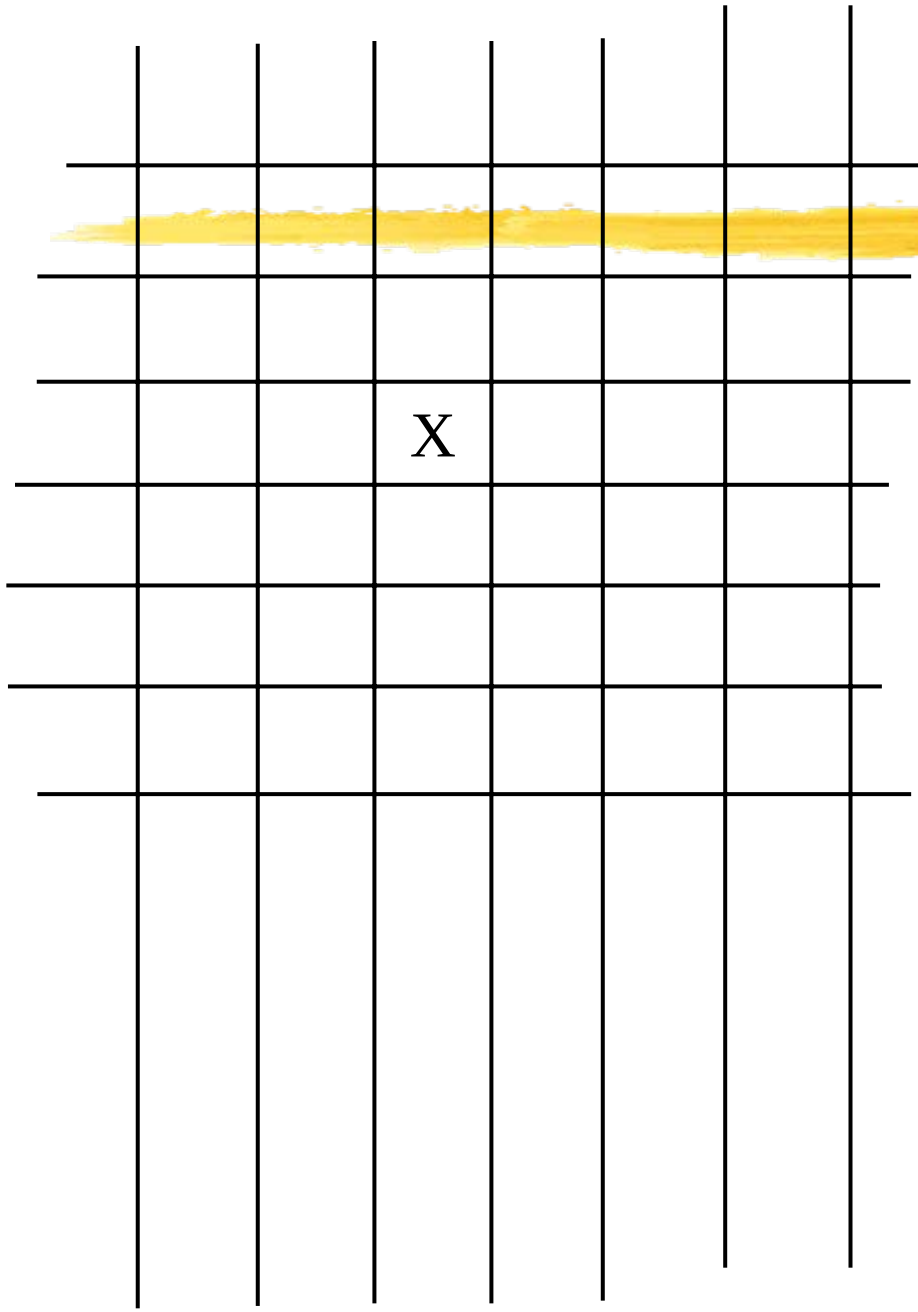
Example Dispatch Policies

- EMCM: Expected Modified Center of Mass
  - Do the conditional expected travel time calculation correctly, conditioned on the centroid of the atom containing the customer

Are fixed preference policies optimal?

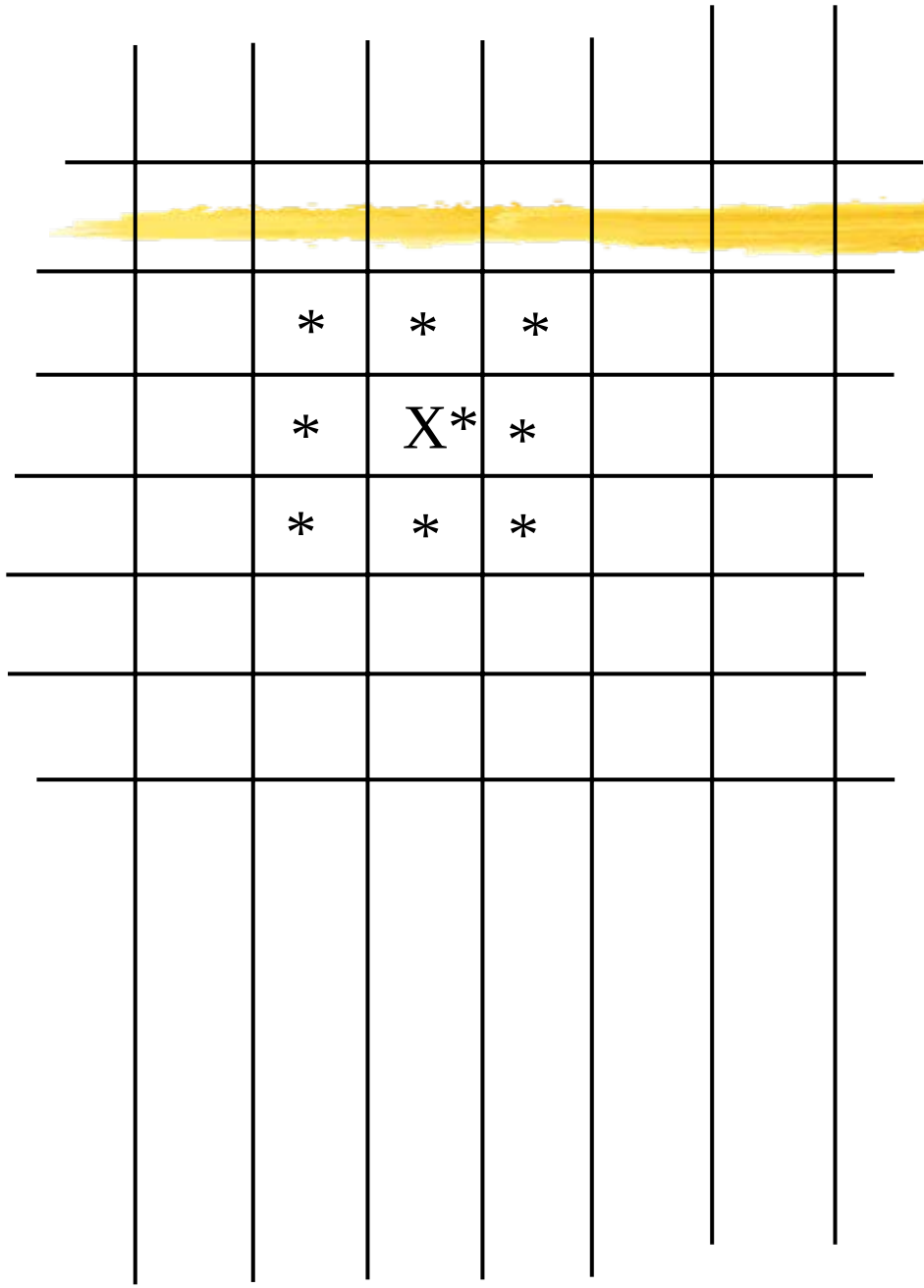
- AVL: Automatic Vehicle Location: dispatch the real time nearest server
  - This can be incorporated into the Hypercube framework, but very carefully!
  - Consider two servers:

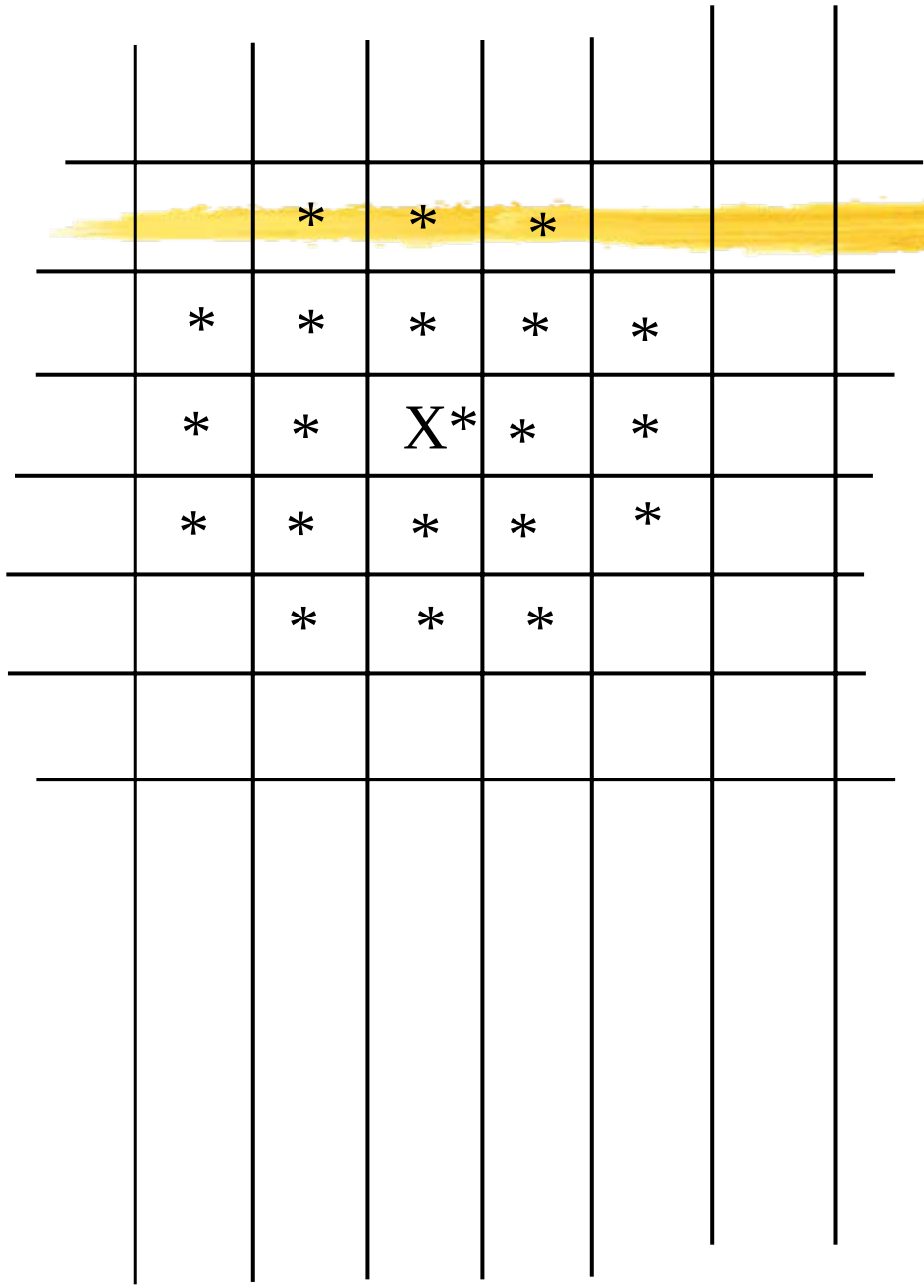




Customer in square marked X. Place an asterisk in each square that could have the closest server.

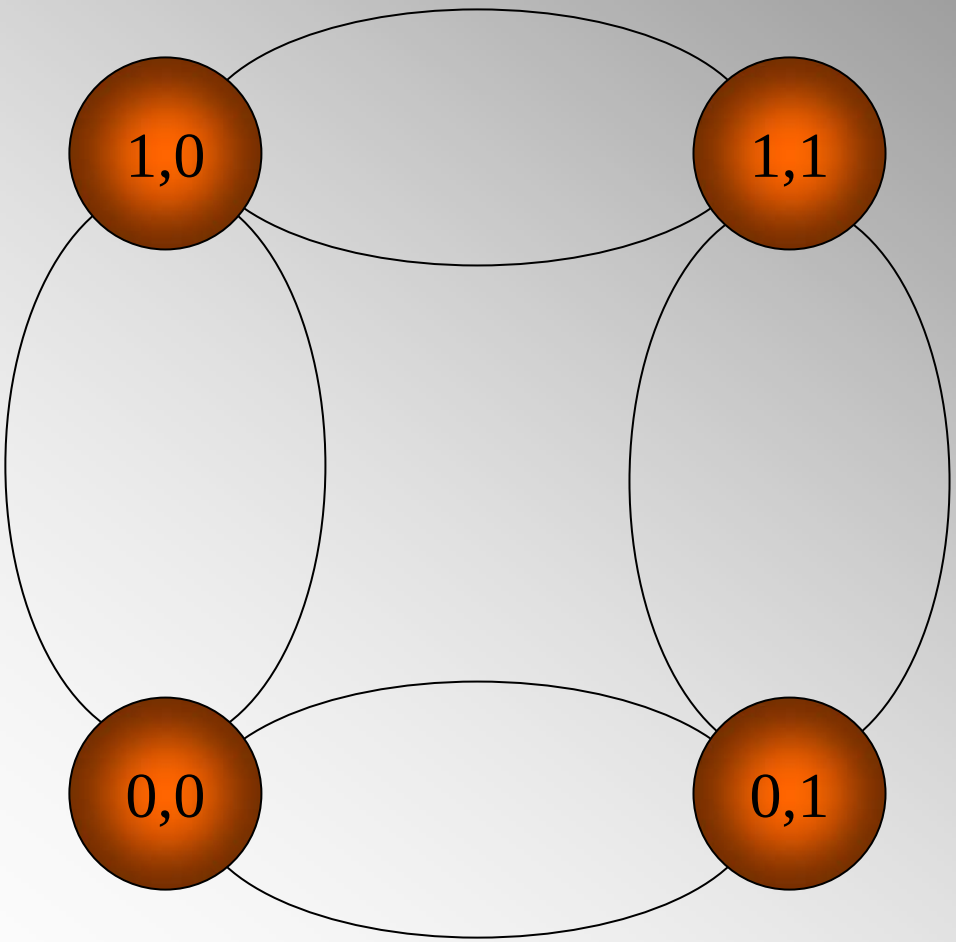
Assume each server is available and is located 'somewhere' in his/her square "police beat."

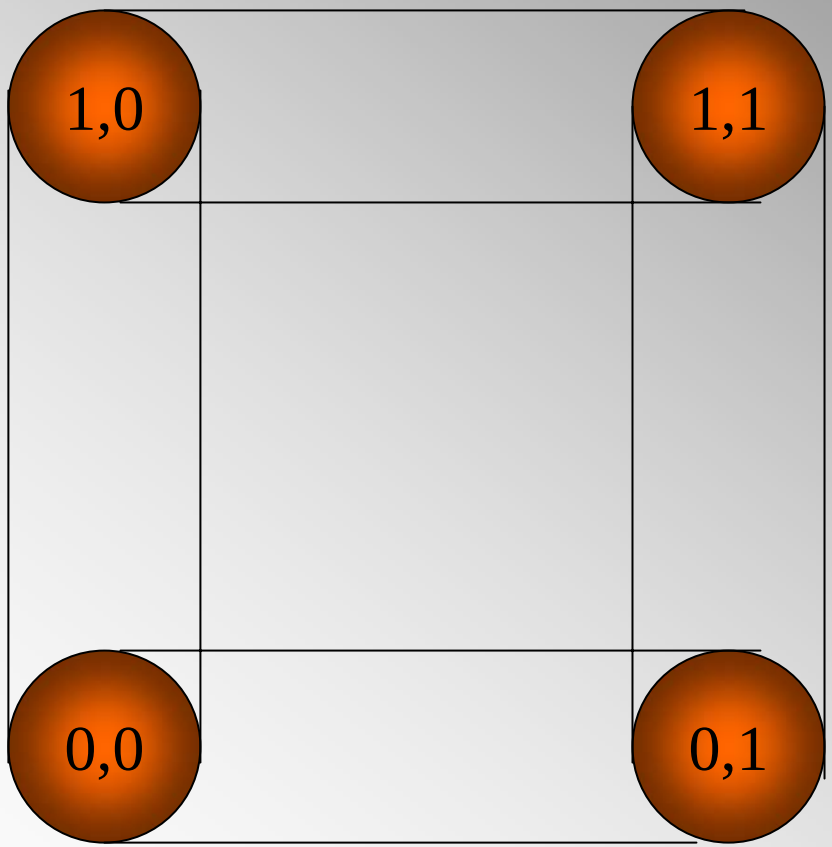

*
*
*
*
X*
*
*
*
*

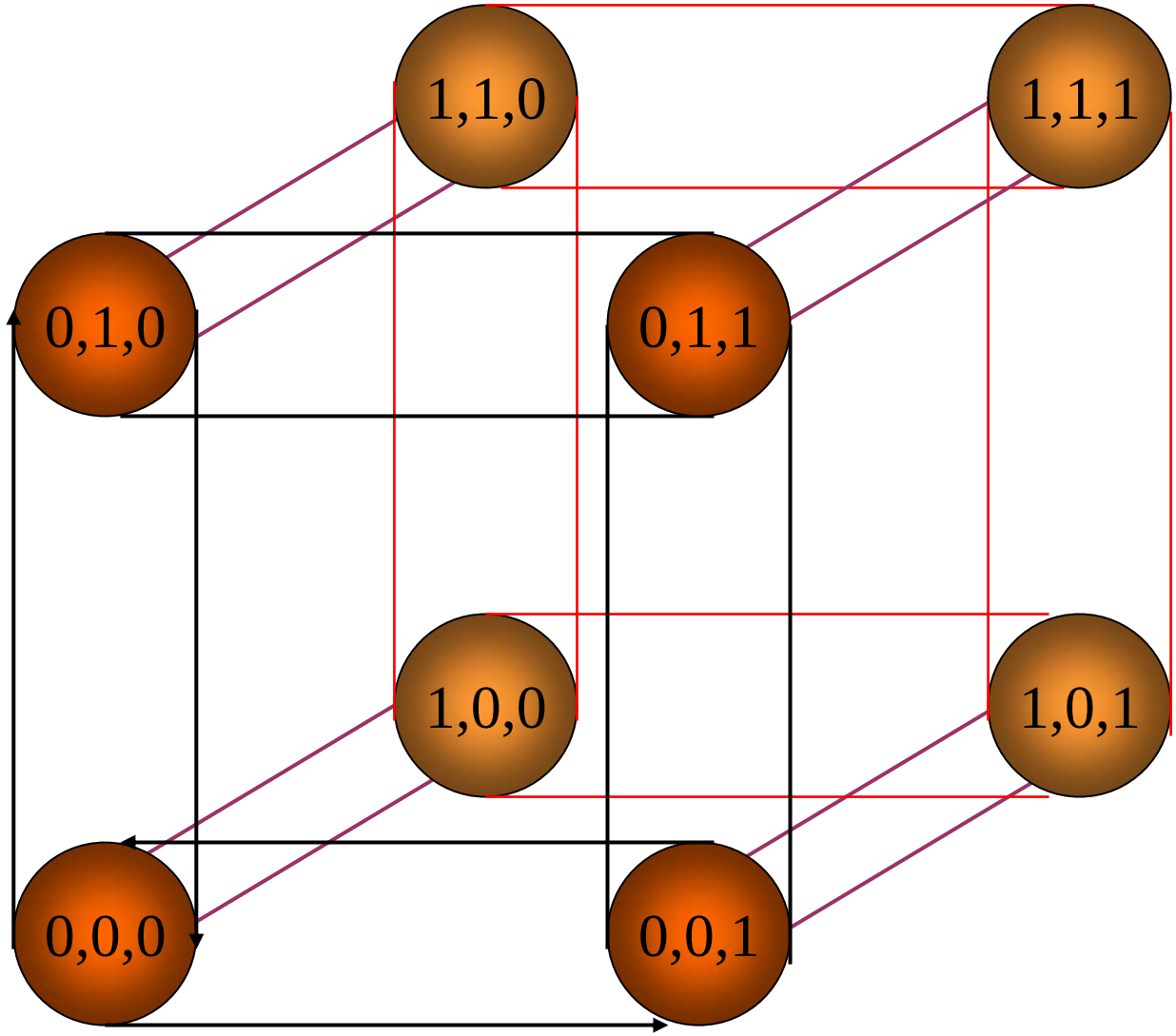

*
*
*
*
*
*
*
*
*
X*
*
*
*
*
*
*
*
*
*
*
*
*

What to know about the Hypercube Queueing Model

- Know the 2-server setup
- Be able to work with a 3-server model
  - Read in the text the formulas to apply
- Forget the cases for $N>3$ servers.
- Know Hypercube Approximation Procedure (still to come -- fasten your seat belts!)


1,0
1,1
0,0
0,1


1,0
1,1
0,0
0,1


1,1,0
1,1,1
0,1,0
0,1,1
1,0,0
1,0,1
0,0,0
0,0,1

Hypercube Approximation Procedure: A General Technique

- Want to reduce dramatically the number of simultaneous equations to solve
- The procedure reduces the number of equations from 2^N simultaneous linear equations to N simultaneous nonlinear equations.
- We look at only those performance measures we need, not at micro-structure of the binary state space

Hypercube Approximation Procedure
A General Technique

Theory: Sampling Servers Without Replacement from *M/M/N* Queue

From $M/M/N/\infty$ we know the aggregate state probabilities:

$$P\{S_k\} \equiv P_k = N^k \rho^k P_0 / k! \qquad k = 0,1,2,...,N-1$$

$$P\{S_N\} \equiv P_N = N^N \rho^N P_0 / (N![1-\rho])$$

$$P\{S_0\} \equiv P_0 = [\sum_{i=0}^{N-1} N^i \rho^i / i! + N^N \rho^N / (N!\{1-\rho\})]^{-1}$$

The Hypercube Model, when the state space is compressed from its cube in *N* dimensions to a 'line' birth and death process, always reduces to an *M/M/N* queue (assuming service times are not server-specific)

Key expression: $P\{B_1, B_2, \ldots, B_j, F_{j+1}\}$

For our applications, we do not need to know the fine grained binary state probabilities. Rather we need dispatch probabilities and server workloads.

What about 'B-' probability reasoning?

"Flips coins" until first Heads is obtained:

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} \approx \begin{cases} \rho^j (1-\rho) & j = 0,1,2,\ldots,N-1 \\ \rho^N & j = N \end{cases}$$

Incompatible with known state probability P_N

Doesn't include biases.

Let's "Divide and conquer":

$$P\{B_1, B_2, \ldots, B_j, F_{j+1}\} = \sum_{k=0}^{k=N} P\{B_1, B_2, \ldots, B_j, F_{j+1} \mid S_k\} P_k \qquad (*)$$

Working carefully and slowly to find the state-conditioned dispatch probabilities:

$$P\{B_1, B_2, ..., B_j, F_{j+1} \mid S_k\} = P\{F_{j+1} \mid B_1, B_2, ..., B_j, S_k\} ... P\{B_2 \mid B_1 S_k\} P\{B_1 \mid S_k\}$$

$$P\{B_1, B_2, ..., B_j, F_{j+1} \mid S_k\} = \frac{N-k}{N-j} ... \quad ... \frac{k-1}{N-1} ..... \frac{k}{N} \quad (**)$$

Can plug (**) back into (*) and obtain an exact expression. Manipulate it to obtain a convenient form as "B-" probability reasoning with an 'A+" correction term:

$$P\{B_1, B_2, ..., B_j, F_{j+1}\} = Q(N, \rho, j) \rho^j (1-\rho) \quad (***)$$

"Correction factor"

Explore properties of Correction Factor

The desired dispatch probabilities can be written as a telescoped expression:

$$P\{B_1, B_2, ..., B_j, F_{j+1}\} = P\{F_{j+1} \mid B_1 B_2 ... B_j\} P\{B_j \mid B_1 B_2 ... B_{j-1}\} ... P\{B_1\}$$

Use above in Eq.(***) to obtain:

$$Q(N, r, j) = [\frac{P\{F_{j+1} \mid B_1 ... B_j\}}{1-\rho}][\frac{P\{B_j \mid B_1 ... B_{j-1}\}}{\rho}]...[\frac{P\{B_1\}}{\rho}]$$

$$\leq 1 \qquad \geq 1 \qquad = 1$$

$G_n^k \equiv$ set of geographical atoms for which unit n is the k^{th} preferred dispatch alternative

$n_{lj} \equiv$ id # of the j^{th} preferred unit for atom l

Set $\mu = 1$

$$\rho_n = \sum_{j \in G_n^1} \lambda_j P\{F_n\} + \sum_{j \in G_n^2} \lambda_j P\{B_{n_{j1}} F_n\} + \sum_{j \in G_n^3} \lambda_j P\{B_{n_{j1}} B_{n_{j2}} F_n\} + \ldots + \lambda P_N / N$$

$$\rho_n = \sum_{j \in G_n^1} \lambda_j (1 - \rho_n) + \sum_{j \in G_n^2} \lambda_j Q(N, \rho, 1) \rho_{n_{j1}} (1 - \rho_n) +$$

$$\sum_{j \in G_n^3} \lambda_j Q(N, \rho, 2) \rho_{n_{j1}} \rho_{n_{j2}} (1 - \rho_n) + \ldots + \lambda P_N / N$$

The last equation gives N nonlinear simultaneous equations in the unknown workloads, ρ_n, *subject to the constraint that*

$$\sum_{n=1}^{N} \rho_n = \lambda \quad \text{"normalization"}$$

Typically converges in 3 to 5 iterations, within 1 to 2% of 'exact Hypercube' results

Response patterns:

$$f_{n_{kj}\,k} = \frac{\lambda_k}{\lambda} Q(N, \rho, j-1)\{\prod_{l=1}^{j-1} \rho_{n_{kl}}\}(1-\rho_{n_{kj}})$$

id # of j^{th} preferred unit for atom k

j-*1* more preferred units

j^{th} preferred unit

Square Root Laws (approximations)

In Chapter 3 we found

$$E[D] = C\sqrt{\frac{A}{N_0}}$$

Area of service region

Number of mobile servers

depended on distance metric
and location strategy

Assumes all N_0 servers are available or free (not busy)

Now consider N to be a R.V.

Might we expect the following to be true?

$$E[D \mid N = k] = C\sqrt{\frac{A}{k}} \qquad k = 1, 2, \ldots, N_0$$

What if the locations of servers were determined by a homogenous spatial Poisson process, with busy servers selected by "random erasers"?

Getting to Expected Travel Distance

$$E[D] = P_0 D_0 + \sum_{k=1}^{N_0} P_k C \sqrt{\frac{A}{k}}$$

From $M/M/N_0$ queueing model

where P_k = Probability of k servers ***available*** ***($M/M/N_0$)***

Moving to E[*D*]

Since $P_0 \cong 0$, we can write

$$E[D] \cong C\sqrt{A}E_{\substack{states\ of \\ M/M/N_0}} \quad [1/\sqrt{N}]$$

We now apply "B-" probability reasoning, to get

$$E[D] \cong C\sqrt{\frac{A}{E[N]}}$$

(Jensen's Inequality shows that this Eq. is a lower bound to true E[*D*].)

Finishing

$$E[N] \approx N_0 - N_0\rho = N_0(1-\rho)$$

$$E[D] \approx C\sqrt{\frac{A}{N_0(1-\rho)}}$$

$$E[T] \approx \frac{C}{v}\sqrt{\frac{A}{N_0(1-\rho)}} + \frac{v}{a}$$

Acceleration term

Great results in practice

Jensen's Inequality

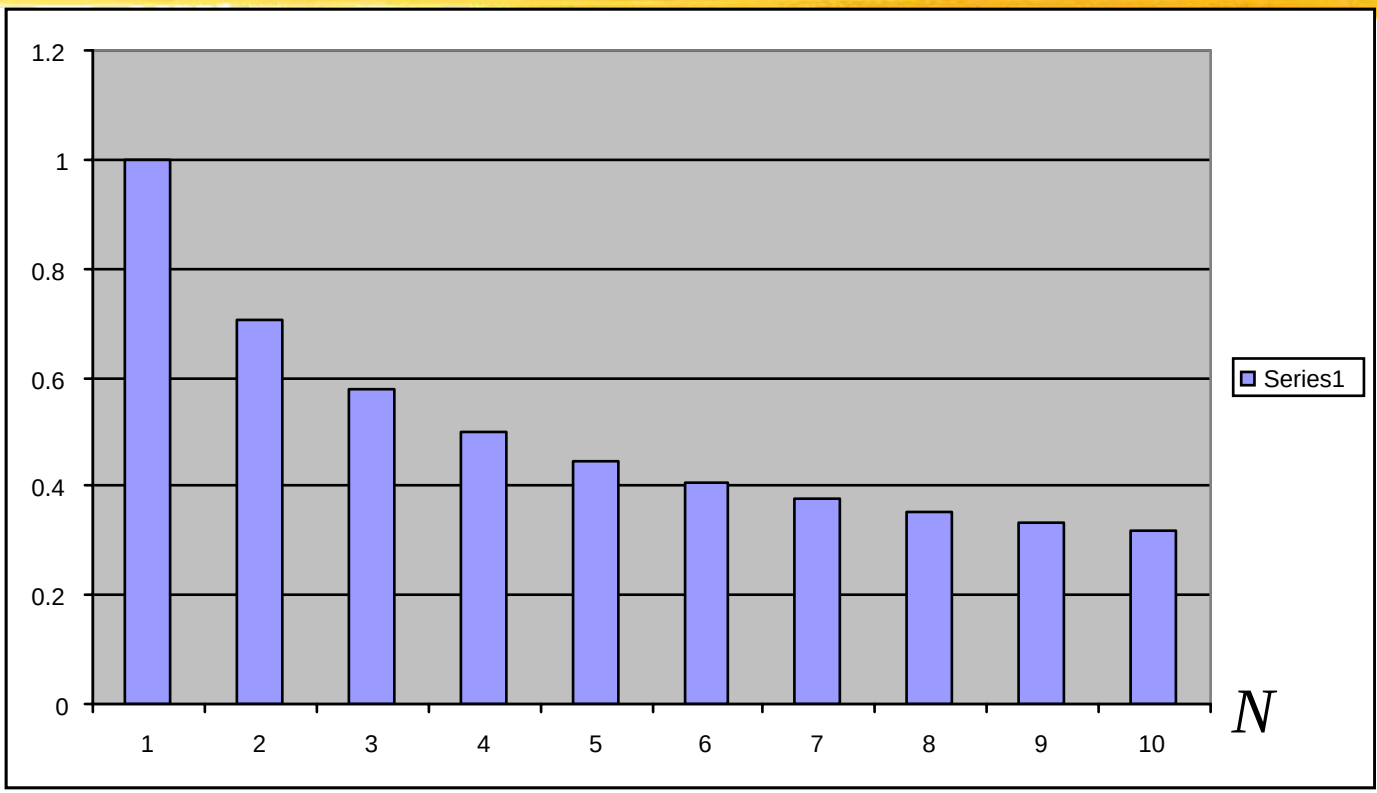
If g(X) is a convex function over the region of non-zero probability, then

$$E[g(X)] \geq g(E[X])$$

(Problem 5.5 explores this further.)

Jensen's Inequality

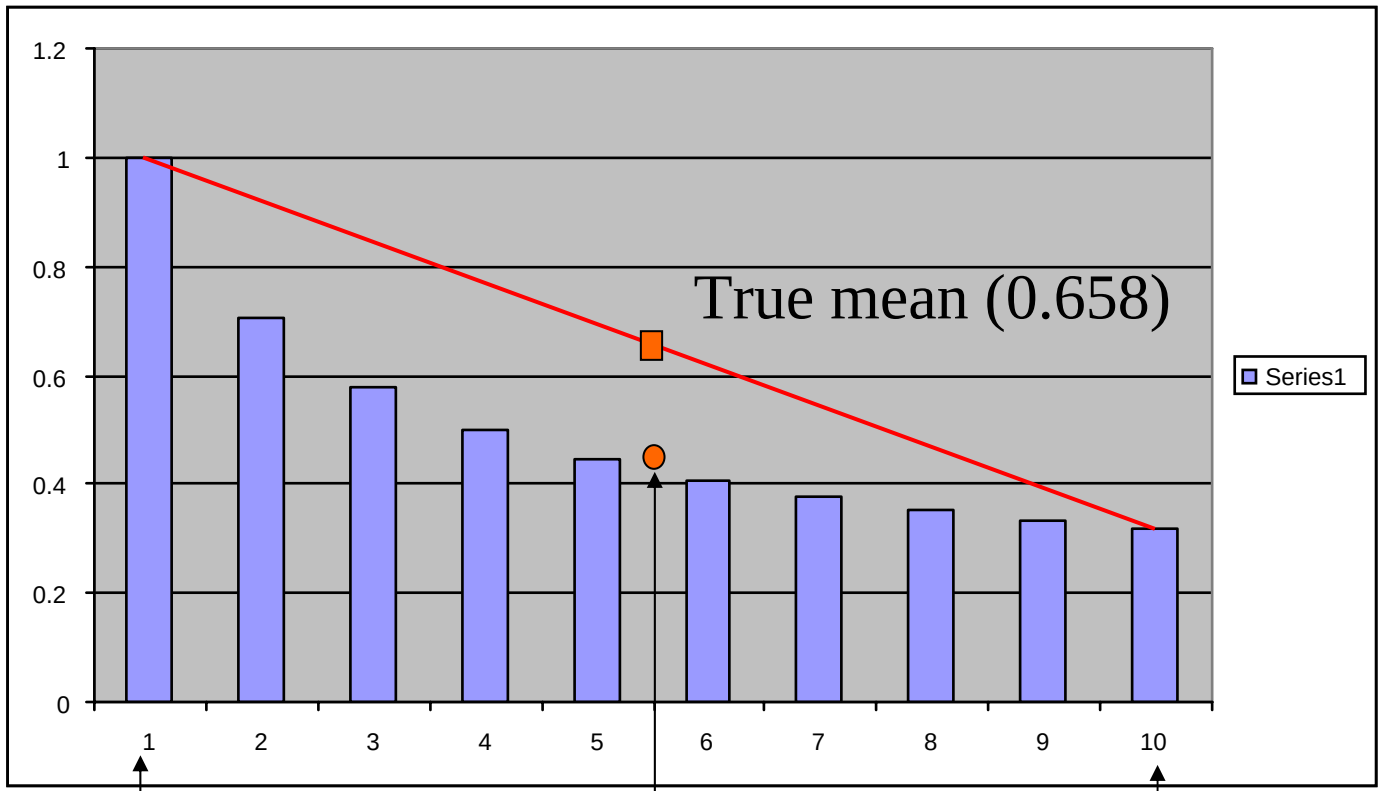
$1/\sqrt{N}$



$E[1/\sqrt{N}] = 0.5*(1) + 0.5*(10)^{-0.5} = 0.5*(1 + 0.316) = 0.658$

$1/E[N]^{-0.5} = 1/(1*0.5 + 10*0.5)^{-0.5} = 1/(5.5)^{-0.5} = 1/2.345 = 0.426$

$1/\sqrt{N}$



Suppose 50% of probability mass here

"B-" mean (0.426)

And suppose 50% of probability mass here