# Computer Algorithms in Systems Engineering

# Spring 2010

# Problem Set 8: Nonlinear optimization: Bus system design

# Due: 12 noon, Wednesday, May 12, 2010

## Problem statement

The analytical equations in lecture 23 are approximate solutions of a system of nonlinear equations. In this homework we will obtain numerical solutions of these equations.

We use Cartesian coordinates. We have a set of parallel bus routes operating in a region of uniform density. All trips are bound to or originate from a point beyond the region (typically a downtown area) that the bus routes serve. The bus routes are spaced a distance g apart, operate at a headway h and charge a fare f. Users are uniformly distributed in the area of dimension X times Y, and walk in a perpendicular direction to the nearest bus route. We ignore bus stop spacing along the routes, the dimensions of the street grid, etc. All of those issues can be handled but make the model more complex.

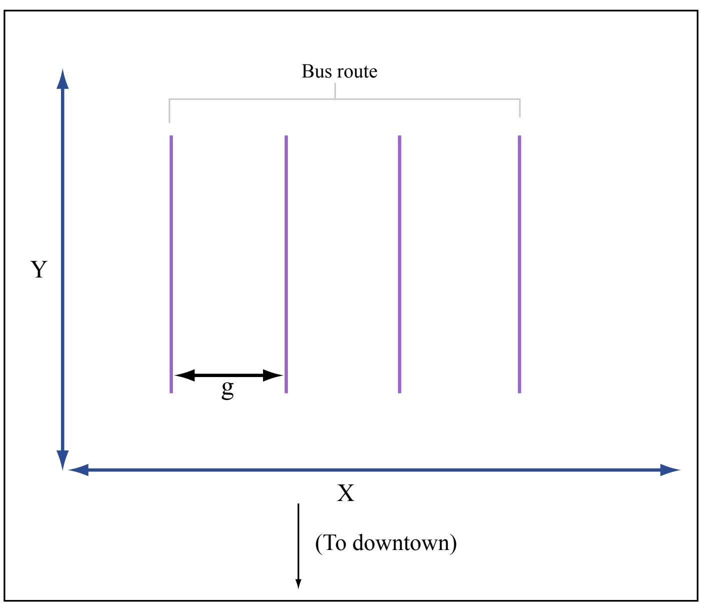


Figure by MIT OpenCourseWare.

To maximize profit, we maximize the difference of revenues minus costs.

Revenue= $TpXYf(a_0-a_2(kh+g/(4j))-a_4f)$

Cost= $2XTcY/(ghv)$

| Variable name | Value | Definition | Units |
|---|---|---|---|
| $p$ | 3.59 | Trip density | Trips/mi$^2$/day |
| $j$ | 0.05 | Walk speed | Miles/minute |
| $k$ | 0.4 | Wait/headway ratio | |
| $c$ | 50 | Bus operating cost | Cents/minute |
| $T$ | 1050 | Length of day | Minutes |
| $V$ | 0.167 | Bus speed | Miles/minute |
| $a_0$ | 0.41 | Bus market share if service equal to auto | |
| $a_2$ | 0.0081 | Bus wait time coefficient | |
| $a_4$ | 0.0014 | Bus fare coefficient | |
| $X$ | 4.0 | Width of analysis area | Miles |
| $Y$ | 6.0 | Length of analysis area | Miles |

The total number of trips by all modes of transport (bus, auto, etc.) is TpXY, or the trip density times the area and the time period. The number of bus trips is the bus market share times the total trips; the bus market share is $a_0-a_2(kh+g/(4j))-a_4f)$. This is a linear approximation to a logit demand model that estimates the market share as a function of the headway h, the route spacing g (which determines the average walking distance to the bus route), and the fare f. The bus revenue is the number of bus trips times the fare f. Note the demand coefficients $a_2$ and $a_4$ are positive; the negative signs for these are in the revenue and demand equations.

The cost of the bus service is derived as: There are X/g routes, each operating T/h trips, each taking Y/v minutes to complete, multiplied by a round trip factor (2) and the operating cost per minute c.

The fare f is in cents. The headway h is in minutes between bus departures. The route spacing g is in miles between routes.

In this homework we will find the optimal values of route spacing g, headway h and fare f. To do so, we can take the derivative of the profit (deficit) function Q with respect to g, h and f.

$$Q = TpXYf(a_0-a_2(kh+g/(4j))-a_4f) - 2XTcY/(ghv) \qquad (0)$$

$$\partial Q/\partial g = -TpXYfa_2/(4j) + 2XTcY/(vhg^2) = 0 \qquad (1)$$

$$\partial Q/\partial h = -TpXYfa_2k + 2XTcY/(vgh^2) = 0 \qquad (2)$$

$$\partial Q/\partial f = TpXY(a_0-a_2(kh+g/(4j))-2a_4f) = 0 \qquad (3)$$

Note that T, X and Y drop out of all the equations. Also note that equations (1) and (2) are very similar and yield a linear relationship between route spacing g and headway h:

$$h = g/(4jk) \qquad (4)$$

Use (4) to eliminate h from (1):

$\partial Q/\partial g = -pfa_2/(4j) + 8jkc/(vg^3) = 0$ (5)

Use (4) to eliminate h from (3):

$\partial Q/\partial f = a_0 - a_2 g/(2j) - 2a_4 f = 0$ (6)

We can solve (5) and (6) analytically, approximately, to obtain:

$g = (64j^2 ka_4 c/ (pva_0 a_2))^{1/3}$ (7)

$f = a0/(2a_4) – (kca_2^2)/(ja_4^2 pva_0))^{1/3}$ (8)

By using (4) we can also find an approximate solution for headway h:

$h = (a_4 c/ (pa_0 a_2 k^2 jv))^{1/3}$ (9)

The approximate solutions (7)-(9) have errors. They are usable for initial analysis, but if exact answers are needed, we may need to solve equations (1)-(3) numerically. Since (2) and (3) are linearly dependent, we only need to solve equations (5) and (6), and can then use (4) to obtain the optimal headway h.

The system of two nonlinear equations which are the first order conditions is:

$-pfa_2/(4j) + 8jkc/(vg^3) = 0$ (10)

$a_0 - a_2 g/(2j) - 2a_4 f = 0$ (11)

Let the equations be numbered 0 and 1. Let the variables g be $x_0$ and f be $x_1$. Rewrite (10) and (11) using $x_0$ and $x_1$, and simplify coefficients:

$f_0(x_0, x_1) = -0.25pa_2 x_1/j + 8jkc/(vx_0^3) = 0$ (12)

$f_1(x_0, x_1) = a_0 - 0.5a_2 x_0/j - 2a_4 x_1 = 0$ (13)

## Assignment

**Do three of the following four parts. You may do the fourth for extra credit.**

1. Use Newton's method to solve the system (12) and (13). Newton's method requires the derivatives of each equation with respect to each variable.

   $\partial f_0/\partial x_0 = -24jkc/(vx_0^4)$ (14)

   $\partial f_0/\partial x_1 = -0.25pa_2/j$ (15)

   $\partial f_1/\partial x_0 = -0.5a_2/j$ (16)

$\partial f_1/\partial x_1 = -2a_4$ (17)

The equations (12) and (13) can be solved with Newton's method, using the derivatives (14)-(17). You will need a good first guess.

2. Use the amoeba to solve this system. The amoeba requires only the function (equation 0) to be maximized. (Amoeba.java from lecture notes minimizes, so reverse signs.) You will need a good first guess.

3. Implement the convex combinations example from lecture 18 in Java. You don't have a linear programming module to use, but minimizing the linear objective function with only two variables and simple constraints can be implemented by looking at the signs of the coefficients in the objective function, and selecting whether to put the variable at its upper or lower bound. Implement the line search rather than using the closed form expression for alpha. You may find it helpful to do this step before doing part 4, since you will have a worked-out test case to follow.

4. Use convex combinations (lecture 18) to solve the bus route/fare problem, with the constraints that the maximum fare f must be \$2 or less and the maximum spacing between routes g must be 1 mile or less. Considerations:
    a. The expression for $\partial z/\partial \alpha$ is messy but not significantly worse than the lecture 18 example from Sheffi. One term in the expression will have (y[0]-x[0]) in the numerator and denominator; don't evaluate this term when y[0]=x[0].
    b. Plot (or just print out) the objective function surface first to understand its shape. It is well behaved except near g= 0, where you should not go.
    c. This is a maximization problem, not a minimization problem like part 3. Be careful to change signs and/or inequalities in the "linear program" and line search methods so you're looking for a maximum and not a minimum.

## Turn In

1. Place a comment with your full name, Stellar username, and assignment number at the beginning of all `.java` files in your solution.
2. Place all of the files in your solution in a single zip file.
    a. Do not turn in electronic or printed copies of compiled byte code (`.class` files) or backup source code (`.java~` files)
    b. Do not turn in printed copies of your solution.
3. Submit this single zip file on the 1.204 Web site under the appropriate problem set number. For directions see **How To: Submit Homework** on the 1.204 Web site.
4. Your solution is due at noon. Your uploaded files should have a timestamp of no later than noon on the due date.
5. After you submit your solution, please recheck that you submitted your .java file. If you submitted your .class file, you will receive **zero credit.**

## Penalties

- 30 points off if you turn in your problem set after Wednesday (May 12) noon but before noon on Friday (May 14). You have two no-penalty two-day (48 hrs) late submissions per term.
- No credit if you turn in your problem set after noon on Friday.

## Substitution for Quiz 2

- This homework may be substituted for quiz 2, or you may take quiz 2. We will ask on Monday May 10 how many of you will be taking quiz 2 to know how many copies to print. If you change your mind after that point, please email me.
- If you submit this homework, it will be counted equally with the other 7 homework sets, all of them making up a total of 85 of the 100 total points for the term. Quiz 1 will be the remaining 15 points.

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