

## msLumped-Parameter (zero-dimensions!) groundwater model of Bangladesh

Your goal in this problem set is to develop a ‘bucket’ model for the hydrology of our study site in Bangladesh. Using this model, you will investigate how the development of dry-season irrigated agriculture in Bangladesh may have affected its groundwater hydrology.

*Figure 1* is a conceptual representation of our study site. We are interested in modeling the hydrology of the sandy aquifer, which underlies our entire study site and is confined by a relatively impermeable clay layer. The fraction of the aquifer area covered by each of the ponds ( $f_p$ ), rice fields ( $f_{ag}$ ), and rivers ( $f_r$ ) is:

$$f_p=0.1$$

$$f_r=0.02$$

$$f_{ag}=0.65$$

$H_a$ ,  $H_p$ ,  $H_r$  refer to the heads of the aquifer, pond, and river respectively.

In this model, you can assume that trees at our site have shallow roots that do not extend all the way into the aquifer. So, by reference to *Figure 1*,  $ET_{tree}$ , which is the flux out of the aquifer due tree transpiration, is equal to zero.

The coupled equations for the rates of change of  $H_a$  and  $H_p$  are:

$$\frac{dH_p}{dt} = k_p * (H_a - H_p) - E_p \quad \text{Equation 1}$$

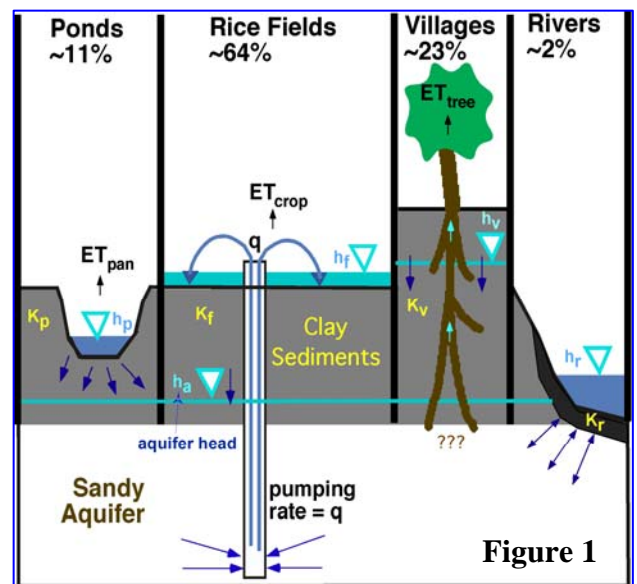
$$S * \frac{dH_a}{dt} = f_p * k_p * (H_p - H_a) + f_r * k_r * (H_r - H_a) - q_i \quad \text{Equation 2}$$

In *Equation 1*,  $E_p$  is the rate of evaporation from the pond per unit pond area. In *Equation 2*,  $q_i$  is the net flux out of the aquifer resulting from the irrigation of rice fields, per unit aquifer area. The following questions will help you develop models for  $E_p$  and  $q_i$ .

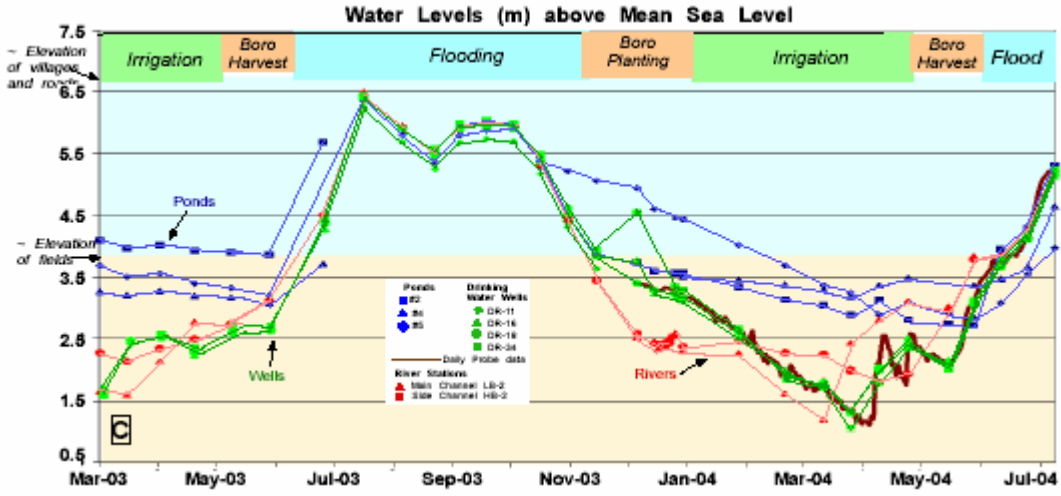
The sediments between the pond and aquifer have a conductance of  $k_p=0.01 \text{ day}^{-1}$ . The sediments accumulated at the river bottom have a higher conductance of  $k_r=0.1 \text{ day}^{-1}$ . The storativity ( $S$ ) of the

aquifer is  $S=0.01$ . The units of  $\frac{dH_p}{dt}$  and  $\frac{dH_a}{dt}$  are

[Length/Time].



The heads in the rivers at our site are controlled by the water levels in the Ganges river, and are not affected by the aquifer head. So, the evolution of  $H_r$  during the dry season is independent of  $H_a$ . Field data for  $H_r$  as a function of time are plotted in red in *Figure 2*.



**Figure 2:** Data on Aquifer (green and brown), River (red), and Pond (blue) heads at our study site.

Assume that the only significant flows into and out of the aquifer occur during the dry season, since during the rainy season (also referred to as the monsoon season or the flooding season), the heads of the aquifer, rivers and ponds are equal so that no flows occur between them.

Your goal is to model the evolution of  $H_a$  and  $H_p$  over the duration of the dry season. Based on Figure 2, the dry season extends approximately between November 1<sup>st</sup> and May 31<sup>st</sup>. At its beginning, following the flooding season, the heads in the rivers, ponds, and aquifer are all at 4m. The following exercises and questions will guide you through developing your model and understanding your results.

**a)** The river head,  $H_r$ , affects the aquifer head  $H_a$  (see Equation 2). However,  $H_r$  itself isn't affected by  $H_a$  since the Ganges River controls it. We will work with  $H_r$  as a known input into Equation 2. This can be done by specifying the values of  $H_r$  at different times in the dry season based on collected field data (Figure 2). Alternatively, we can use a simplified representation of the  $H_r$  data shown in Figure 2 by modeling  $H_r(t)$  as a Sine function of time  $t$ , so that the resulting function resembles the collected data.

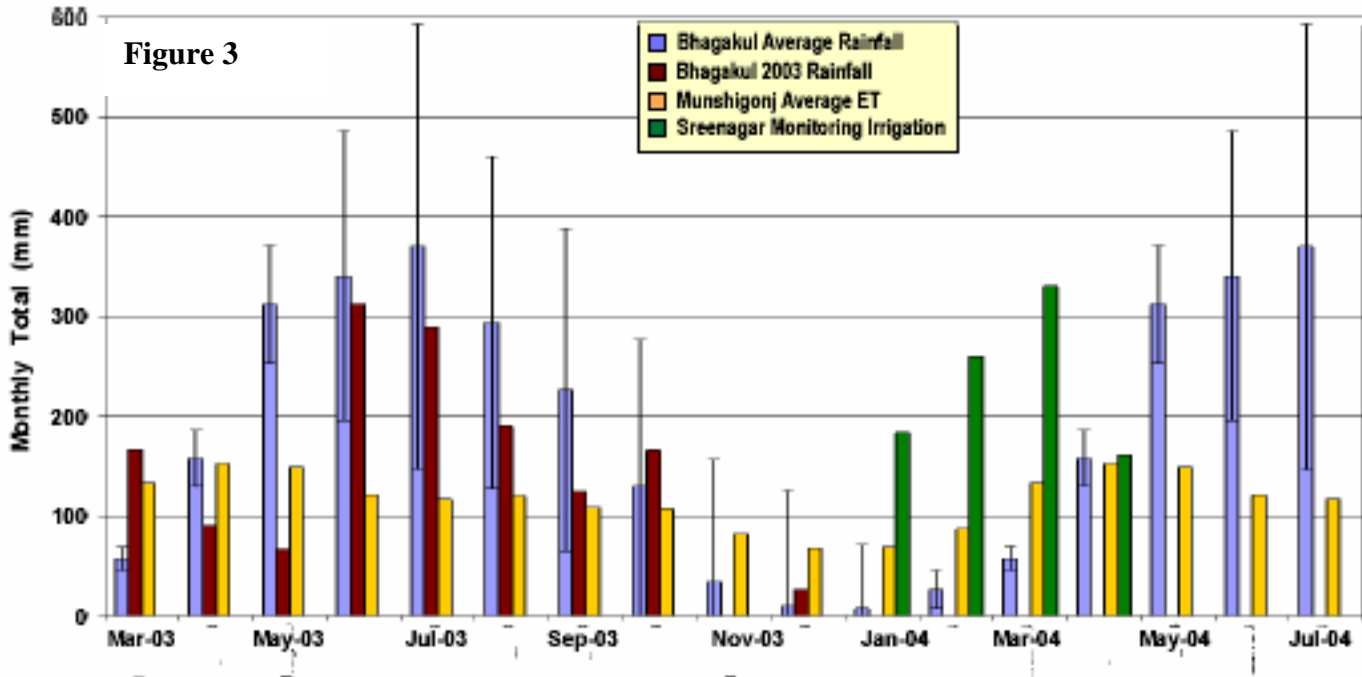
$$H_r = H_{amp} * \sin\left(\frac{t}{tspan} * \pi + \pi\right) + 4 \quad \text{Equation 3 (in meters)}$$

$H_{amp}$  is the amplitude of  $H_r(t)$ ,  $H_{amp} = 2.5$  m  
 $tspan$  is the duration of the dry season.

1. Check that this equation for  $H_r(t)$  gives appropriate values of  $H_r$  for  $t=0$ ,  $t=tspan$ , and  $t=0.5*(tspan)$ .
2. Show that another equivalent equation for  $H_r(t)$  is  $H_r = -H_{amp} * \sin\left(\frac{t}{tspan} * \pi\right) + 4$  (in meters).

**b)** Evaporation at our study site can be modeled with reference to the evaporation at the nearest meteorological station in Munshigonj (with the crystal ball). Figure 3 shows the estimated evaporation at Munshigonj, which will be referred to as the reference evaporation ( $E_o$ ).  $E_o$  is usually

expressed as a flow *per unit area of the evaporating surface* with units of [Length/Time]. In *Figure 3*, the units of  $E_o$  are mm/month.



$E_o$  data is collected regularly and reliably, so it is useful to estimate evaporation at our site in relation to  $E_o$ . The actual evaporation rate of the ponds at our site (per unit pond area) can be modeled as

$$E_p = C_p * E_o$$

$C_p$  is a coefficient estimated empirically, by collecting field data on the evaporation rates of the ponds and relating them to  $E_o$ .  $C_p$  was found to be 1.4.

Similarly, the actual transpiration rate of rice plants in the agricultural fields at our site (per unit field area) can be modeled as  $E_{ag} = C_{ag} * E_o$ .  $C_{ag}$  was found to be 1.1.

- Using data from *Figure 3*, estimate the mean value of  $E_o$  during the dry season. You will need to convert your result into units of meters/day, since it will be an input into your equations for  $dH_a/dt$  and  $dH_p/dt$ .

c) Assume that the farmers pump out water from the aquifer to irrigate their rice fields at a rate  $p$ . The rice crop transpires some of the water applied to the fields, and the rest percolates back through the clay layer into the aquifer.

1- If you ignore the lag between the times when water is applied to the fields and when it reaches the aquifer, what is the net flux out of the aquifer that results from irrigation (labeled  $q_i$  in the equation for  $dH_a/dt$ )? Use the equation for  $E_{ag}$  in part (b) to express  $q_i$  in terms of known quantities. Make sure you express it *per unit aquifer area*, since it will be one of the fluxes in *Equation 2* that determine  $dH_a/dt$ .

2- *Figure 2* shows that the farmers do not irrigate their crops during the entire dry season. Write a simple model for  $q_i(t)$ , using your answer to part (c) and information on the start and end of the irrigating period from *Figure 2*.

d) Code the system of equations for  $dH_a/dt$  and  $dH_p/dt$  into a Matlab *function M-file*, which defines your differential equations and is an input file for the *ode45* function. The aquifer head and pond head should be the only unknowns in your equations, in addition to your independent variable  $t$ . Make sure you define all other variables and parameters you include in your equations. Solve for your aquifer and pond heads using the *ode45* function. *ode45* outputs the solutions ( $H_a(t)$  and  $H_p(t)$ ), as well as the vector of time points ( $t$ ) for which  $H_a$  and  $H_p$  are solved. Click on the button “Workspace” on the left side of your Matlab window to see these outputs (*Figure 4*).

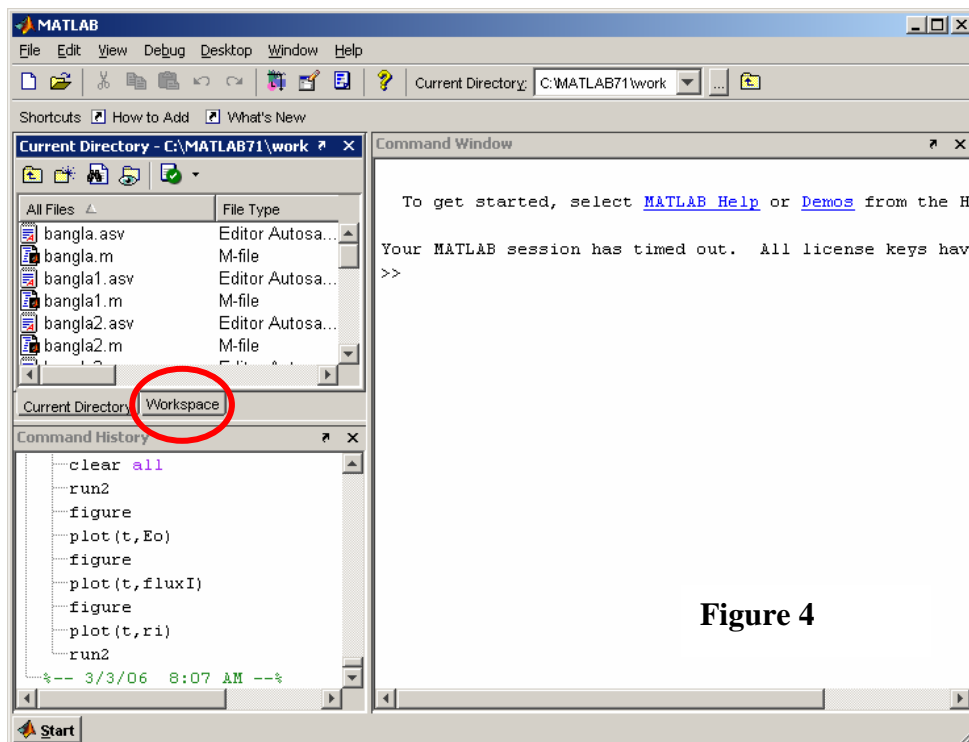


Figure 4

In the Workspace window, double click on the vector of time points and on the matrix of heads to see their values. What are the two columns of the head matrix?

1- Plot the *ode45* solutions for  $H_a(t)$  and  $H_p(t)$ . Figure out how to plot  $H_a(t)$  separately: to retrieve the values for  $H_a(t)$  separately from the head matrix output by *ode45*, you need to specify which rows and which column of the head matrix you're interested in.

2- Overlay on your plot of  $H_a(t)$  and  $H_p(t)$  the plot for river head  $H_r(t)$ . To do this, you will need to re-define the equation for  $H_r(t)$  at the command prompt, even though you've defined it in your initial *function M-file*. This is because any variable or parameter you define in your *function M-file* is internal to that function. The only outputs into your Workspace are the *ode45* solutions, i.e. your aquifer and pond heads and the vector of time points  $t$ . Confirm that once you define  $H_r$  at the command prompt, it shows up in your Workspace. How many rows does the  $H_r$  vector have? Why? To overlay the  $H_r(t)$  curve on top of your plot for  $H_a(t)$  and  $H_p(t)$ , type *hold on* at the command prompt before entering the *plot(t, Hr)* command.

e) Interpret your resulting plot. How does the onset of irrigation affect  $H_a(t)$ ,  $H_p(t)$  and  $H_r(t)$ ? Compare your plot to the data presented in Figure 2.

f) Think about how you might improve your model so that it better reflects the conditions at our study site. As you elaborate your model, run it and plot the solutions for  $H_a(t)$  and  $H_p(t)$ , then overlay the curve for  $H_r(t)$ . See if your modifications to the model produce results that agree better with the data in *Figure 2*. Below are a couple of suggestions for improving your model. The first suggested modification is required, the second is optional, and you can come up with other possible modifications. If you have an idea for improving your model, Hanan can help you with the coding.

**1-** In part *b*, you specified  $E_o$  to be a constant value equal to the average value of  $E_o$  over the dry season. Based on the data in *Figure 3*, formulate a better model for  $E_o(t)$  and code it into your *function M-file*.

**2- (OPTIONAL, but produces nice results)** The  $q_i$  flux you described in (*c-1*) is the maximum flux when all the rice fields are being irrigated. However, this maximum flux isn't maintained during the whole irrigating period, since many fields are irrigated for only a portion of this period. To account for this, include a factor  $r_i$  in your equation for  $dH_a/dt$ , which multiplies  $q_i$  and specifies the fraction of agricultural area that is actually irrigated at any time  $t$ . Define  $r_i(t)$  as a Sine function of  $t$ , varying from 0.5 at the start of the irrigating period, to 1 in the middle of this period, and back to 0.5 at its end. This model for  $r_i(t)$  implies that half the rice fields are actually irrigated initially, and this fraction increases with time until all the fields are being irrigated, and then proceeds to decline back to 0.5 at the end of the irrigating period.  $r_i(t)$  must be specified as zero outside the irrigating period. Refer to (*c-2*) for information on the extent of the irrigating period, and to part (*a*) to remind yourself how to model a variable as a Sine function of time.

g) So far, you've been looking at the solutions for the aquifer and pond heads. These are evolving in time due to the fluxes between the different water bodies. In the next few questions you will be investigating these fluxes.

**1-** Define the equation for  $FluxP(t)$ , the flux between the aquifer and the pond per unit area of aquifer. In your equation, when is the result for  $FluxP$  positive, and would that imply recharge to or discharge from the aquifer? If needed, adjust the signs in your equation so that a positive  $FluxP$  indicates recharge to the aquifer, i.e. flow from the pond to the aquifer.

**2-** Similarly, define the equation for  $FluxR(t)$ , the flux between the aquifer and the river per unit area of aquifer. In your equation, when is  $FluxR$  positive, and would that imply recharge to or discharge from the aquifer? If needed, adjust the signs in your equation so that a positive  $FluxR$  indicates recharge to the aquifer.

**3-** Create a new Matlab *M-file*, in which you code the equations for  $FluxP(t)$  and  $FluxR(t)$ . Make sure you name the variables defined by these equations:  $FluxP$  and  $FluxR$  (you'll soon see why this is necessary). You will be running this script after you run *ode45*, so it will use the solutions for  $H_a(t)$  and  $H_p(t)$  that *ode45* produces. As in part (*d-1*), to retrieve the solutions for  $H_a(t)$  separately, you need to specify which rows and which column of your output head matrix you're interested in. Similarly for  $H_p(t)$ . Your script will also use the vector of  $t$  values (time points) produced by *ode45*. Remember that you need to define any other parameters or variables used in your equations for  $FluxP(t)$  and  $FluxR(t)$  (such as  $H_r$ ,  $k_p$ ,  $k_r$ ,  $f_p$  and  $f_r$ ), even though you've defined them in your initial *function M-file*. This is because these variables and parameters are internal to your function, and so aren't saved by Matlab into your Workspace when you run *ode45* (this is explained in part *d*).

**4-** The net flux out of the aquifer due to irrigation, per unit aquifer area, is  $FluxI(t) = -r_i(t) * q_i(t)$  or simply  $-q_i(t)$ , depending on whether or not you completed the optional part (f-2). Add the equation for  $FluxI(t)$  to your new *M-file*. Name the variable defined by this equation: FluxI. Remember you will need to define any variable or parameter you use in this equation that is not an output of *ode45*. Use the models for  $q_i(t)$  and  $E_o(t)$  that you created in parts (c-2) and (f-1), respectively. Since  $q_i(t)$  and  $E_o(t)$  have different values depending on the time period (for example,  $q_i(t)$  is zero outside of the irrigating period), you need a “for loop” that specifies these variables for every time point in the  $t$  vector output by *ode45*. Instructions to help you write the code for  $FluxI(t)$  are available on *stellar* under “Materials”, in the document “FluxI.doc”. Does your equation produce positive or negative values for  $FluxI(t)$ ? Does  $FluxI(t)$  constitute discharge from or recharge to the aquifer?

i) Run your new script by typing its filename at the command prompt, to obtain solutions for  $FluxP(t)$ ,  $FluxR(t)$ , and  $FluxI(t)$ . Remember that  $FluxP(t)$  and  $FluxR(t)$  are functions of  $H_a(t)$  and  $H_p(t)$ , so you need to have run *ode45* first to obtain the solutions for the aquifer and pond heads. Plot these fluxes and interpret your results. Note especially changes in sign in  $FluxP(t)$  and  $FluxR(t)$ . What do they mean? Are they consistent with your plot for  $H_a(t)$ ,  $H_p(t)$  and  $H_r(t)$ ? Also interpret changes in the slopes of these flux curves.

j) From the “Materials” section of the *stellar* website, download the Matlab script “*TotalFluxes.m*” and save it into your current directory, where you are saving your other scripts. Run it to obtain solutions for the variables: *totaldischargeP*, *totalrechargeP*, *totaldischargeR*, *totalrechargeR*, and *totaldischargeI*. The *TotalFluxes.m* script uses your results for  $FluxP(t)$ ,  $FluxR(t)$ , and  $FluxI(t)$ . That is why you had to give specific names to the vectors containing the solutions for these fluxes. The *TotalFluxes.m* script integrates these fluxes over the duration of the dry season to compute the variables:

*totaldischargeP*: The total discharge from the aquifer to the pond per unit aquifer area over the dry season.

*totalrechargeP*: The total recharge from the pond to the aquifer per unit aquifer area over the dry season.

*totaldischargeR*: The total discharge from the aquifer to the river per unit aquifer area over the dry season.

*totalrechargeR*: The total recharge from the river to the aquifer per unit aquifer area over the dry season.

*totaldischargeI*: The total flow out of the aquifer due to irrigation per unit aquifer area over the dry season.

Once you run the script, Matlab saves these variables and their values into your Workspace.

- Confirm that the variables representing flows out of the aquifer have negative values, while those representing flows into the aquifer have positive values.

**1-** What is the total flow *out of* the aquifer per unit aquifer area over the dry season? What is the total flow *into* the aquifer per unit aquifer area over the dry season?

**2-** What is the net change in aquifer head over the dry season (refer to your plot for  $H_a$ ,  $H_p$ , and  $H_r$  or plot  $H_a(t)$  again)? Given that  $S=0.01$ , what change in volume of water per unit aquifer area does this correspond to?

**3-** Based on your answers to (j-1) and (j-2): does your model for the aquifer head conserve mass?

**k)** The aquifer at our study site has a depth  $D=100\text{m}$ , and average porosity  $\Theta = 0.25$  (25%).

**1-** What is the volume of water contained in the aquifer?

**2-** Assume that over a year, the only flows into and out of the aquifer at our site occur during the dry season. Using your answers to (j-1) and (k-1), calculate the residence time of water in the aquifer. If you find that the total outflows from and inflows into the aquifer are not equivalent, take their average.

**l)** Use your model to figure out how the aquifer's water budget was changed as a result of the development of irrigated agriculture in Bangladesh.

**1-** Adjust your model so that no water is extracted from the aquifer for irrigation. You can do this by changing the value of  $f_{ag}$  to 0 in your initial *function M-file* that defines your system of equations for  $dH_a/dt$  and  $dH_p/dt$ .

**2-** Run *ode45*, and plot  $H_a(t)$  and  $H_p(t)$ . Overlay the plot of  $H_r(t)$ . Compare your plot for  $H_a(t)$ ,  $H_p(t)$  and  $H_r(t)$  in the absence irrigated agriculture to the plot of these heads that you produced when you accounted for irrigation in your model.

**3-** Run the *M-file* script you wrote that solves for  $FluxP(t)$ ,  $FluxR(t)$ , and  $FluxI(t)$ . Plot these fluxes. Again, compare these results, in the absence of irrigation, to the results of the model in which you accounted for irrigation.

**4-** Run the script you downloaded, "*TotalFluxes.m*". Compute the residence time of water in the aquifer in the absence of pumping for irrigation. Compare it to the residence time that you computed in (k-2).