MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 15 Multiobjective Optimization and Utility
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Multiobjective problems

Benefits and costs are often **incommensurate** (measured in different units) are they may accrue to **different parties** (equity issues):

Examples:

Benefits ⟷	**Costs**
Hydropower output (MWhrs, $)	Loss of species habitats or recreational opportunities (Units ???)
Additional crop revenues for upstream farmers benefiting from a water diversion ($)	Reduced crop revenues for downstream farmers with less water ($)
Information provided by a field monitoring program (Units ??)	Sampling cost ($)

Multiobjective analysis recognizes this by revealing **tradeoffs** among different objectives.

Extension of the crop allocation example

Extend previous example by considering 2 objectives – **maximization of crop revenue** and **minimization of pesticide concentration in groundwater**:

Decision variables:

x_1 = mass of Crop 1 grown (tonnes = 10^3 kg)

x_2 = mass of Crop 2 grown (tonnes = 10^3 kg)

$$\underset{x_1,x_2}{Maximize} \quad 6x_1 + 11x_2 \qquad \text{Crop revenue (\$)}$$

$$\underset{x_1,x_2}{Minimize} \quad 5x_1 + 2x_2 \qquad \text{Pesticide concentration in groundwater (ppm)}$$

such that :

$$\begin{array}{rl}
2x_1 + x_2 \leq 104 & \text{Water constraint } (10^3 \text{ m}^3\text{/season}) \\
x_1 + 2x_2 \leq 76 & \text{Land constraint (ha)} \\
-x_1 - x_2 \leq -25 & \text{Minimum production constraint (tonnes)} \\
-x_1 \leq 0 & x_1 \text{ non-negativity constraint} \\
-x_2 \leq 0 & x_2 \text{ non-negativity constraint}
\end{array}$$

All constraints and the feasible region are the same as before.

It is convenient to transform the problem so that **both objectives are maximized**. Call the negative of pesticide concentration "environmental quality":

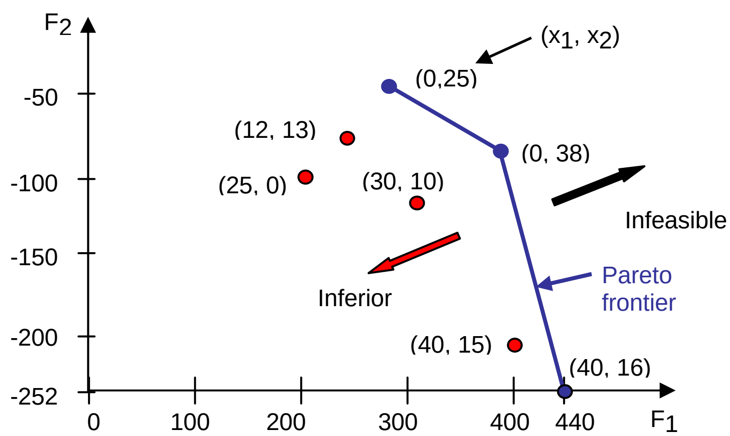
$$\underset{x_1,x_2}{Maximize} \quad F_1(x_1,x_2) = 6x_1 + 11x_2 \qquad \text{Crop revenue (\$)}$$

$$\underset{x_1,x_2}{Maximize} \quad F_2(x_1,x_2) = -5x_1 - 2x_2 \qquad \text{Environmental quality (-ppm)}$$

that :

$$\begin{array}{rl}
2x_1 + x_2 \leq 104 & \text{Water constraint } (10^3 \text{ m}^3\text{/season}) \\
x_1 + 2x_2 \leq 76 & \text{Land constraint (ha)} \\
-x_1 - x_2 \leq -25 & \text{Minimum production constraint (tonnes)} \\
-x_1 \leq 0 & x_1 \text{ non-negativity constraint} \\
-x_2 \leq 0 & x_2 \text{ non-negativity constraint}
\end{array}$$

There is a **tradeoff** between the revenue and environmental quality objectives : As x_1 and/or x_2 increases crop revenue increases environmental quality decreases (and vice versa)



The nature of the tradeoff is revealed in plot of F_2 vs F_1:

- Each **feasible solution** corresponds to a single point in the F_2 - F_1 plane.
- If a solution is **inferior** it is possible to increase one of the objectives without decreasing the other.
- **Non-inferior (Pareto optimal)** solutions lie on the **Pareto frontier** which forms a boundary separating inferior and infeasible solutions.
- Different Pareto optimal solutions represent different **tradeoffs** between the two objectives – if one objective is increased by moving to another Pareto solution the other objective cannot increase (and usually decreases).

How can we identify the Pareto frontier in general?
Best alternative is usually to carry out a **parametric analysis**:

- Treat all but one objective (F_i, i =,2,…N) in an N-objective problem as constraints with specified right-hand values for $F_2,\ldots, F_N$.
- Maximize the remaining objective F_1. As the right-hand side values $F_2,\ldots, F_N$ are changed the solutions trace out the Pareto frontier.

In the example, **treat crop production objective as a constraint and maximize environmental quality** F_2 as a function of F_1**:**

$$\underset{x_1, x_2}{\mathit{Maximize}} \quad F_2(x_1, x_2) = -5x_1 - 2x_2$$

such that :

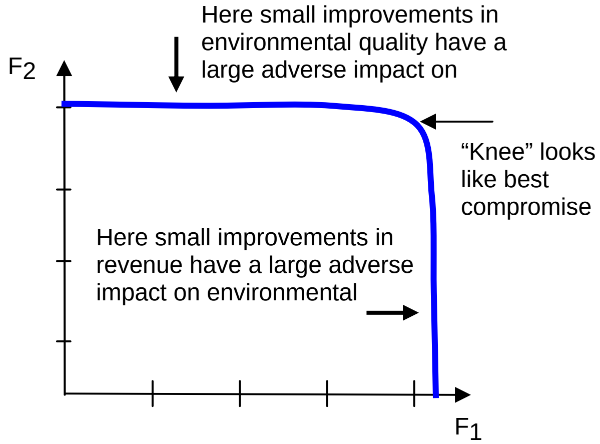
$$\begin{array}{rl}
6x_1 + 11x_2 \geq F_1 & \text{Crop production must be at least } F_1 \\
2x_1 + x_2 \leq 104 & \text{Water constraint } (10^3 \text{ m}^3\text{/season}) \\
x_1 + 2x_2 \leq 76 & \text{Land constraint (ha)} \\
-x_1 - x_2 \leq -25 & \text{Minimum production constraint (tonnes)} \\
-x_1 \leq 0 & x_1 \text{ non-negativity constraint} \\
-x_2 \leq 0 & x_2 \text{ non-negativity constraint}
\end{array}$$

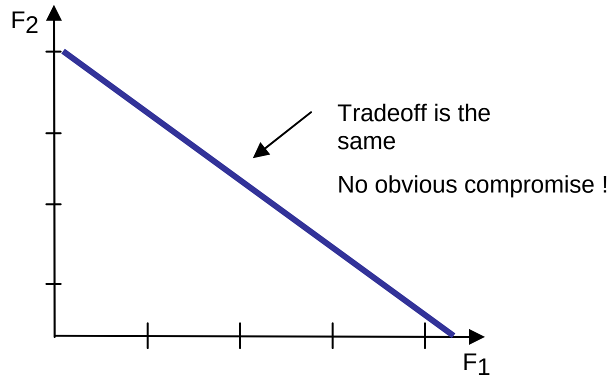
The Pareto frontier can be obtained in GAMS by solving the above problem in a loop which varies F_1 from 275 (the minimum feasible Pareto value) to 440 (the maximum feasible Pareto value).

Same result is obtained if we **treat environmental quality as a constraint and maximize crop production** F_1 as a function of F_2.

Above concepts apply equally well to nonlinear and discrete multi-objective optimization problems.

Different types of tradeoffs:

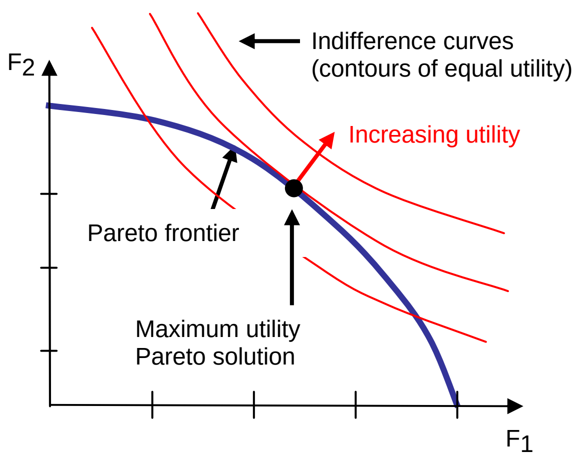




Utility

Tradeoff curves do not tell us **which** Pareto optimal solution to adopt.
One approach for finding a single optimum solution is to identify a **utility (or preference)** function.

The utility function defines combinations of $F_1, F_2, \ldots, F_N$ values that a particular party (individual, group, etc.) finds **equally acceptable**. Contours of constant utility are called **indifference curves**.



Pareto curve can be viewed as an **equality constraint** in a new optimization problem where we seek to **maximize utility**. Then maximum utility solution lies at the point where the gradients to the utility function and Pareto frontier constraint point in the same direction.

Utility functions are difficult to measure, although economists have developed indirect ways to estimate them from surveys.

A typical example of a two-objective utility function $U(F_1, F_2)$ that may be fit to survey data is the Cobbs-Douglas function:

$U(F_1, F_2) = F_1^{\alpha} F_2^{\beta}$ where α and β are specified (or fit) non-negative coefficients

The dependence of the utility function on any given objective value is typically nonlinear.

Utility and Risk

For the crop allocation example, consider the dependence of utility on revenue F_1 for fixed environmental quality F_2.

To examine effects of uncertain F_1 expand $U(F_1)$ in a Taylor series around mean revenue $\overline{F_1}$:

$$U(F_1) = U(\overline{F_1}) + \frac{\partial U}{\partial F_1}(F_1 - \overline{F_1}) + \frac{1}{2}\frac{\partial^2 U}{\partial F_1^2}(F_1 - \overline{F_1})^2 + \ldots$$

Mean of this expression is:

$$\overline{U(F_1)} = U(\overline{F_1}) + \frac{1}{2}\frac{\partial^2 U}{\partial F_1^2}\sigma_{F_1}^2 + \ldots \quad \text{where } \sigma_{F_1}^2 = \text{variance of } F_1$$

When there is **no uncertainty**: $\sigma_{F_1}^2 = 0 \rightarrow \overline{U(F_1)} = U(\overline{F_1})$.

When there is **uncertainty**: $\sigma_{F_1}^2 > 0 \rightarrow$ relationship between $\overline{U(F_1)}$ and $U(\overline{F_1})$ depends on sign of $\partial^2 U / \partial F_1^2$.

Three possibilities:

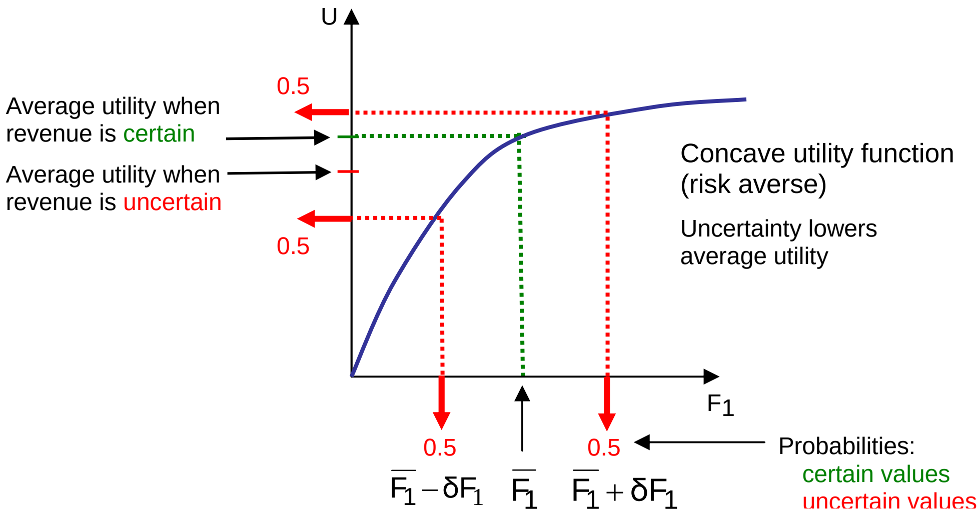
- **Risk averse**: $U(F_1)$ is **concave**, $\partial^2 U / \partial F_1^2 < 0$ **mean utility is lower** when F_1 is uncertain (risk lowers utility)
- **Risk neutral**: $U(F_1)$ is **linear**, $\partial^2 U / \partial F_1^2 = 0$ **mean utility is the same** when F_1 is uncertain (risk has no effect on utility)
- **Risk seeking**: $U(F_1)$ is **convex**, $\partial^2 U / \partial F_1^2 > 0$ **mean utility is higher** when F_1 is uncertain (risk raises utility).

Utility is often a **concave function of revenue** (decision-maker is **risk averse**) for sufficiently large revenue.

In the crop allocation example this could reflect the fact that the marginal utility gained by having more revenue gradually decreases as environmental quality declines.

Example:
Consider a risk adverse farmer faced with uncertain revenue because of uncertainty in the farm water supply.

F_1 has 2 possible values $\overline{F_1} \pm \delta F_1$, each with probability = 0.5.



Suppose the (concave) utility function for this risk adverse farmer is $U(F_1) = \ln(F_1)$. The farmer can sell a crop option for a price *P* before the growing season starts. The option guarantees the farmer revenue *P*. The actual value of the crop is either $\overline{F_1} + \delta F_1$ or $\overline{F_1} - \delta F_1$, depending on uncertain water availability. What price is the farmer willing to accept for the option?

Suppose $\overline{F_1} = \$\,1000$, $\delta F_1 = \$\ 200$

If farmer sells the option for price *P* the mean (certain) utility is

$$\overline{U(F_1)} = \ln(P)$$

If farmer does not sell the option and accepts risk the mean utility is:

$$\overline{U(F_1)} = 0.5\ln(\overline{F_1} + \delta F) + 0.5\ln(\overline{F_1} - \delta F) = 3.55 + 3.34 = 6.89$$

Equate these two mean utilities and solve for *P*:

$$P = \exp(6.89) = \$\ 982.40$$

So the farmer is willing to sell the crop option for *P* = \$982.40 rather to obtain expected revenue of \$1000. The **risk premium** is \$17.60.

If the farmer is **risk neutral** he would require that *P* = \$1000 and the risk premium would be zero.