12.086/12.586 Problem Set 1
Due: October 2

In this first problem set, we introduce a couple of neat ideas about random walks that might come up in models (e.g. your final projects). It's possible to do almost everything here with paper, pencil, and a couple cups of coffee. In fact, a couple of these only require you to give a reasonable, intuitive argument. So feel free to use any method you like. Problem 4 will require you to run a script I've written for you in MATLAB.®

-Robert

1. **Return Probability**

   Loris are small primates which live in South East Asia, Sri Lanka, and India (Figure 1). One particular group of them feed on nectar from certain flowers that grow in the canopy. The problem is that these flowers are distributed far apart and once a loris feeds on a flower it takes a while for the flower to produce more nectar. Thus an individual often wastes a lot of energy to visit an empty flower.

   (a) We are interested in the amount of stress an individual Loris puts on the total number of flowers. One could conceive of similar problems facing organisms that live in a number of different dimensions (e.g. a nematode grazing on a flat surface), so generalize the problem to d dimensions. Make the simplifying assumption grazers scour the area randomly (unbiased random walk).

      i. How does the volume V of likely positions visited by the grazer scale in time?
      ii. How does the density ρ of grazed sites scale in time? Discuss the result in $d =$1,2, and 3 in the limit that time goes to infinity. (Hint: How does the mean-square distance vary as a function of time?)

Figure 1: Two lorises

   (b) Many random walks in nature are biased. One of the famous examples is a bacterium moving up a chemical gradient. A bacterium measures the local concentration of the chemical, takes a step in a random direction, and measures the change in the chemical concentration. Depending on the change in concentration, the bacterium adjusts the step size. Thus, the bacterium moves up (or down if it is toxic) the gradient with probability $q = \frac{1}{2} + \epsilon$. Since this is a random process there is some probability that the bacterium will move up the gradient, and then back down to where it started. If we trap

a bacterium in a gradient, what is the probability that it will never return to where it started? Assume that the gradient can be modeled as a one-dimensional lattice (Hint: Let p be the probability that the bacterium does return. Suppose that the bacterium takes a step up the gradient to $x = 1$. It will then either return to $x = 0$ or go a random walk which might eventually take it back to 1. The probability that it returns to $x = 1$ is the same as returning to $x = 0$. Use this to write p in terms of ϵ.)

(c) Optional: Can you think of an example like the first which takes place on a fractal? How could the Loris interact as to avoid visiting empty flowers? (Hint: in real life they do)

2. **First Passage in One Dimension**
In the last problem we focused on "will a random walk ever hit the same place twice?" This time, the question is "how long until a random walker returns to a point?" In the lectures we saw that this question is related to the distribution of river basin sizes; however the problem has many other applications. In one-dimension, the size of a population can be thought to take a random walk (do births outnumber deaths at a given point). The first passage to zero corresponds to the time when the population goes extinct. Brain cells behave similarly. A neuron is constantly being bombarded with chemicals that make it either easier or harder for it to carry a charge. When the charge reaches a certain threshold, the neuron fires. This problem has been considered by Gerstein and Mandelbrot (*Random walk models for the spike activity of a single neuron* Biophysical Society 1964) among others. There are also analogues in higher dimensions (e.g. when will a Loris return to a flower). To make things mathematically simpler we will only consider one-dimensional systems on a lattice. We will find the probability a random walker will return after a time T.

(a) Will constraining ourselves to one-dimension influence the final result?

(b) Will constraining ourselves to a lattice influence the final result?

(c) The code, first_return.m, simulates a random walk and finds the time of first return.
Use this code (or write your own or do the analysis) to generate an *ensemble* of first return times (about 10000 test cases works well). Use this data to estimate the probability of a random walk having a particular return time $F(T)$. In general, $F \sim T^{-\nu/2}$. Estimate ν for the one dimensional case. Where do you suppose the 2 in the exponent come from?
[1]

(d) What is the expected time $\langle T \rangle$ until the walker returns to zero? What does this mean?

3. **Sediment Accumulation**
This problem is concerned with sediment accumulation over geological time scales, i.e., the change in the thickness of a sediment layer as a function of time. Two of the main processes that affect the sediment thickness are deposition and erosion, both of which are complex geomorphic processes whose rates fluctuate in time. Figure 2 illustrates the sediment thickness as a function of time. It has been widely noted that the thickness of a sediment section does not scale linearly with the time interval it represents (Figure 3). These data suggest that the thickness $\delta(t)$ scales like t^α, with $\alpha \simeq 0.5$. Equivalently, the sedimentation rate $R(t) = \delta(t)/t$ scales like t^β, with $\beta = \alpha - 1 \simeq -0.5$. That is, the sedimentation rate appear to be slowing

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[1]If you are using MATLAB once you have an ensemble of times and you want a histogram, use [freq,x]=hist(X), and get ν by fitting $\log(freq) \propto -\nu/2 \, \log(T)$. (Hint: bin the first return time logarithmically, and remember to multiply by the Jacobian to get the original PDF if you do)

down with the length of the time interval. In this problem, we will try to construct a simple model that captures this apparent slowdown.

(a) Let $h(n)$ be the height of the sediment after time n. It is important to realize that the ordinate of Figure 3 does not correspond to height $h(n)$, but rather to height difference $\delta(t)$ after time t:

$$\delta(t) = h(n+t) - h(n).$$

We model the sedimentation process as a discrete, 1-D random-walk. At each step, the height of the sediment changes by a random amount $\delta(1) = h(n+1) - h(n)$, with $\delta > 0$ corresponding to net deposition, and $\delta < 0$ to net erosion. Assume that all steps are independent and are drawn from a common probability density function $p(s)$ that has mean μ and variance σ^2.

    i. What is the probability density function $p[\delta(t)]$ for large t?

    ii. What is the average rate of sediment accumulation at time t, for large t?

    iii. Are these results in agreement with the data shown in Figure 3?

(b) The key point is that only *preserved* thickness is observed. That is, when $\delta(t) < 0$ all the material from the corresponding time period has been removed and no stratigraphic record remains. Thus, we need to calculate the apparent thickness changes $\hat{\delta}(t)$ (the observable thickness changes of preserved sediments). Express the pdf $p[\hat{\delta}(t)]$ in terms of the pdf of the height difference, $p(\delta)$.

(c) Show that, for large t, the apparent rate of sediment accumulation $\theta(t)$ is

$$\theta(t) = \mu + \frac{\sigma}{\sqrt{2\pi t}} \left[e^{-\mu^2 t / 2\sigma^2} / \Phi(\mu\sqrt{t}/\sigma) \right],$$

where Φ is the cumulative distribution function (cdf) of the standard normal distribution, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \mathrm{d}y.$$

(d) Under what condition is the rate given above consistent with the data shown in Figure 3?

4. **Diffusion-limited Aggregations**

So far in this problem set, we've considered a few interesting one-dimensional random walk problems, but these systems tend to rely heavily on a lack of particle interaction. While this can often be fruitful in modeling the chemotaxis pathways of 1-dimensional bacteria, or in determining the effect of primates on local fauna development, more complicated systems thrive on interaction. One such theory of interaction is known as diffusion-limited aggregation (DLA) - in this process, an initial "sticky" particle is placed in a system. Other particles are allowed to randomly walk until they hit this first particle, after which they stick to it and begin to form an aggregate. As you may have guessed this phenomenon is widely seen in nature. Examples of this include aggregates of dust in the atmosphere and organic aggregates in the ocean (or 'marine snow').

(a) Vj g"eqfg"FNCdb."qp"yj g"UVWF["OCVGTKCNU"rci g"uses MATLAB to simulate diffusion-limited ci i tgi cvkqp0The simulation places a sticky spot in the center of a 101 by 101 two-dimensional lattice, and lets random walkers in from the outside until they encounter this sticky boundary. Upon 'sticking' to this boundary, they become part of it, and the next walker is released. Run this script, and plot the aggregate.

(b) Write a short code that makes a plot of the mass of the aggregate contained in a circle of radius r as a function of r. How does the mass scale with the radius?

(c) Find the partial differential equation and boundary conditions that correspond to DLA in the continuum limit.
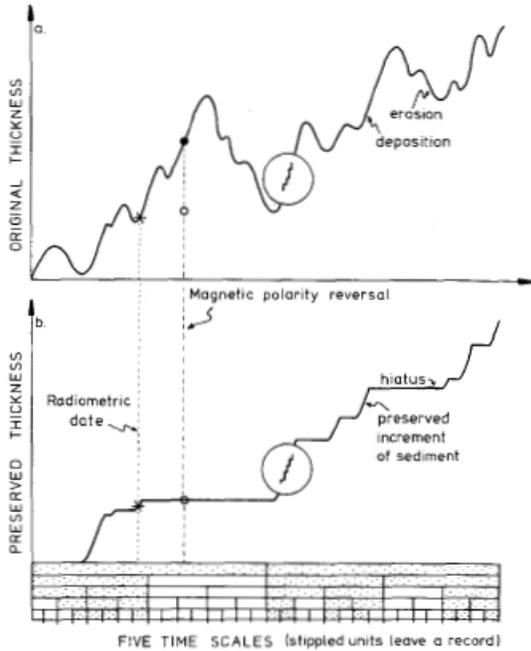
Figure 2: Illustration of the sedimentation process. (a) sediment thickness as a function of time. (b) preserved sediment, observable at the last time point of the process. Bottom bars indicate whether a time interval left a sedimentary record (dotted), or all deposited material from time interval was eroded away (open). From P.M. Sadler, D.J. Strauss, *J Geol Soc London* **147**, 471 (1990).
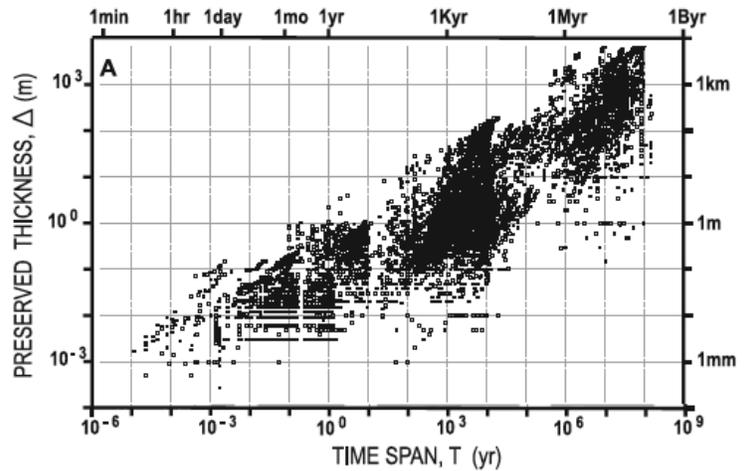
Figure 3: Sedimentation accumulation data. Preserved thickness of section as a function of time span represented by section. From D.J. Jerolmack, P.M. Sadler, *J Geophys Res-Earth* **112**:F3, F03S13 (2007), doi:10.1029/2006JF000555.

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Fall 2014