Chapter 3

Internal Gravity Waves

3.1 Interfacial waves

We have thus far considered the dynamics of the air-water interface. The surface gravity wave motions that this interface permits owe their existence to the restoring force associated with the *density difference* across the interface. [Because we did not consider the effects of motions in the air—we neglected variations in atmospheric pressure—we implicitly assumed that $\rho_{air} \ll \rho_{water}$, which is a very good assumption.] In fact, similar waves are also possible at any *internal* interface in a fluid across which there is a density discontinuity, such as shown in Fig. 3.1. Suppose the densities above and below the interface are ρ_1, ρ_2, respectively (where $\rho_2 > \rho_1$). In general, further complexities are introduced by the different depths of the fluid layers; if we concentrate on layers of equal depth D, then the dispersion relation for the interfacial waves is the same as for the surface wave case (2.20), except that the shallow water wave speed, $\sqrt{gD}$ in the surface wave case, is replaced by

$$c_0 = \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} D} \,. \tag{3.1}$$

[Note that this reduces to $\sqrt{gD}$ in the case $\rho_1 \ll \rho_2$.] The origins of the additional factor are not hard to see. The frequency of any oscillations on the interface depend on the restoring force acting on any deviations of the interface; this force is proportional to the density difference across the interface, $\rho_2 - \rho_1$. The frequency also depends on the *inertia* of the fluid, which is proportional to $\rho_2 + \rho_1$. If the density difference is small ($\rho_2 - \rho_1 \ll \rho_2 + \rho_1$),

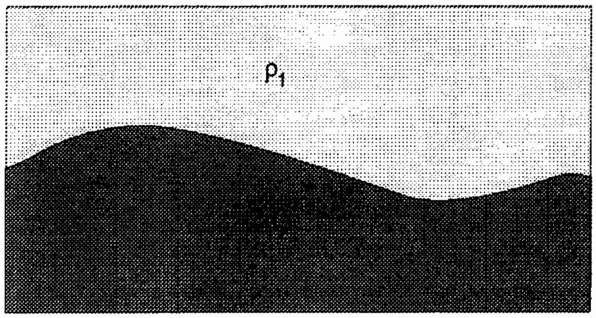

Figure 3.1: Interfacial waves on the interface between two fluids of different density.

the wave speed is much slower than that of surface waves[1].

Consider now the behavior of a fluid with many such layers, as shown in Fig. 3.2. Suppose something makes a disturbance on the surface. When we considered surface waves on deep water, we saw that there are motions within the water, extending a characteristic distance k^{-1} below the surface. If there is a density interface withint this distance, that will be affected by these motions, and will become distorted by them. In turn, this will set up motions in the layer beneath that interface, which will perturb the layer below, etc., etc.. Thus, in addition to propagating horizontally along the interfaces, the disturbance will propagate *vertically* within the fluid. This is unlike the case of surface waves on a fluid of constant density; such internal waves can propagate vertically by virtue of the fluid's internal density structure. This is illustrative of the way fluids can often support three-dimensional wave propagation.

[1] One way of experiencing this is to gently rock a jar of oil-vinegar dressing to set up oscillations on the interface; when you find the resonance, the period will be much longer than if you repeat the experiment with a jar of oil or vinegar alone.

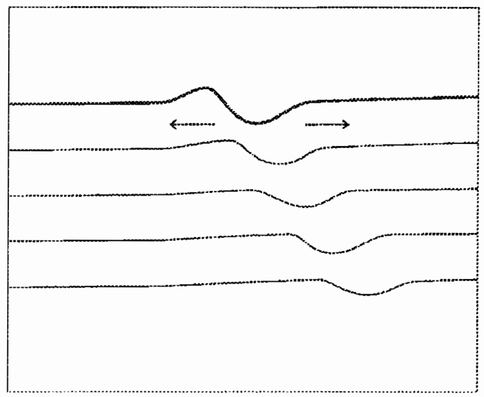

Figure 3.2: Interfacial (internal) waves in a fluid with many constant-density layers.

3.2 Internal waves in a fluid with continuous stratification

Most fluids—including the ocean and the atmosphere, do indeed have internal variations of density. Sometimes these variations occur sharply, but there is almost always a continuous variation of density, which supports internal waves in much the same way. In fact, if (see Fig. 3.3) we compare two fluids, one with many layers of slightly different density (which increases monotonically with depth), and the other with a continuous but otherwise similar density profile, it does not take much imagination to see that they would both behave very similarly; each density profile will support internal waves.

In fact, if ρ varies linearly with z in an incompressible fluid, the dispersion relation for plane waves of the form $w = Re\left[W_o e^{i(kx+ly+mz-\omega t)}\right]$ is

$$\omega = \pm N\sqrt{\frac{k^2+l^2}{k^2+l^2+m^2}}, \tag{3.2}$$

where

$$N = \sqrt{\frac{g}{\rho}\frac{d\rho}{dz}} \tag{3.3}$$

is the "buoyancy frequency". Note that as $m \to 0$, $\omega \to \pm N$ (this actually corresponds to the case where air motions are exactly vertical), and that, in

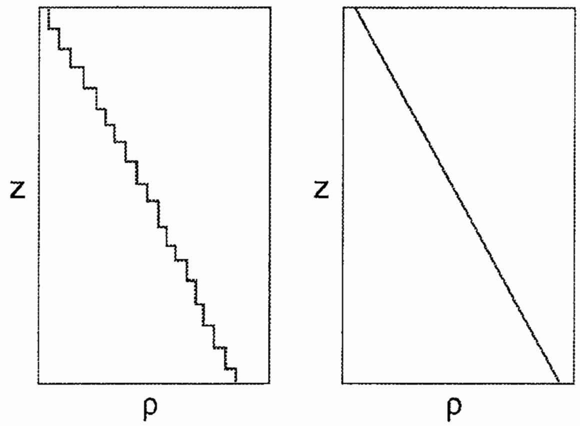

Figure 3.3: Density stratification of a fluid with (left) density steps and (right) continuous stratification.

general, $|\omega| \leq N$, so that the buoyancy frequency is the *maximum* frequency of these waves. The corresponding period, $2\pi/N$, may range from a few minutes in the atmosphere to several minutes to hours or days in the ocean.

3.3 Vertical density structure of the ocean

A typical vertical profile of ocean density is shown in Fig. 3.4. The actual profile at any place and time will vary but the main characteristics are the same:

1. A "mixed layer" with the top few tens of meters, within which the density is almost uniform;

2. A "thermocline" at depths of around 100m, with a sharp density contrast (but note that its magnitude is only a few percent);

3. Below the thermocline, weaker but persistent gradients of density.

Such a profile is capable of supporting fast surface waves, slower interfacial waves on the thermocline, and much slower internal waves in the deep ocean. Internal waves are ubiquitous in the ocean.

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3.4 Gravity waves in the Atmosphere

3.4.1 The vertical structure of a compressible atmosphere

Unlike the ocean, of course, the atmospheric density varies dramatically with height, primarily because of the compressibilty of air. We know that, in most situations (*i.e.*, unless vertical accelerations are significant, which only usually happens for small-scale motions), hydrostatic balance is satisfied:

$$\frac{\partial p}{\partial z} = -g\rho \, . \tag{3.4}$$

To determine how ρ and p vary with z, we need to invoke the **equation of state** (the relationship between pressure, density and temperature). For air, a good representation is the ideal gas law

$$pV = R^{*}T \tag{3.5}$$

where V is the volume of one kilomole of air and $R^{*} = 8314.3$ J deg^{-1}kmol^{-1} is the **universal gas constant.** Since $\rho = M/V$, where $M = 28.97$ kg is the mass of one kilomole of dry air (of mean molecular weight 28.97), eq. (3.5) may be written

$$p = \rho RT \, , \tag{3.6}$$

where $R = R^*/M = 287$ J deg^{-1} kg^{-1} is the **gas constant for air**.

Now, substituting from (3.6) into (3.4), we obtain

$$\frac{\partial p}{\partial z} = -g\frac{p}{RT} = -\frac{p}{H} , \tag{3.7}$$

where

$$H = \frac{RT}{g} \tag{3.8}$$

is the pressure **scale height**. If H is constant (isothermal atmosphere), for example, pressure decays exponentially with height, with e-folding scale H:

$$p = p_s e^{-\frac{z}{H}} , \tag{3.9}$$

where p_s is the surface pressure (at $z = 0$), and density likewise:

$$\rho = \frac{p}{RT} = \frac{p_s}{RT} e^{-\frac{z}{H}} . \tag{3.10}$$

If the atmosphere is not isothermal, but $T = T(z)$, $H = H(z)$ and

$$p = p_s \exp\left(-\int_0^z \frac{dz'}{H(z')}\right) , \tag{3.11}$$

so H is still the measure of the rate of decay of p, but in a local sense. For a typical value of $T = 270K$, $H \simeq 7.9$ km.

An example of a typical atmospheric temperature vs. height profile (at 35°N in April) is shown in Fig. 3.5. Within the troposphere ($z <$ 10km at high latitudes, $z <$ 16km in the tropics), temperature decreases with altitude at a rate of about 7K km^{-1}; in the stratosphere (up to $z \simeq$ 50km), the temperature increases slowly with altitude.

3.5 Potential temperature and static stability

Consider the vertical displacement of air parcel, as shown in Fig. 3.6. The parcel P is displaced from z to $z + dz$, *i.e.*, from p to $p + dp$, where, from (3.4),

$$dp = -g\rho(z)\, dz .$$

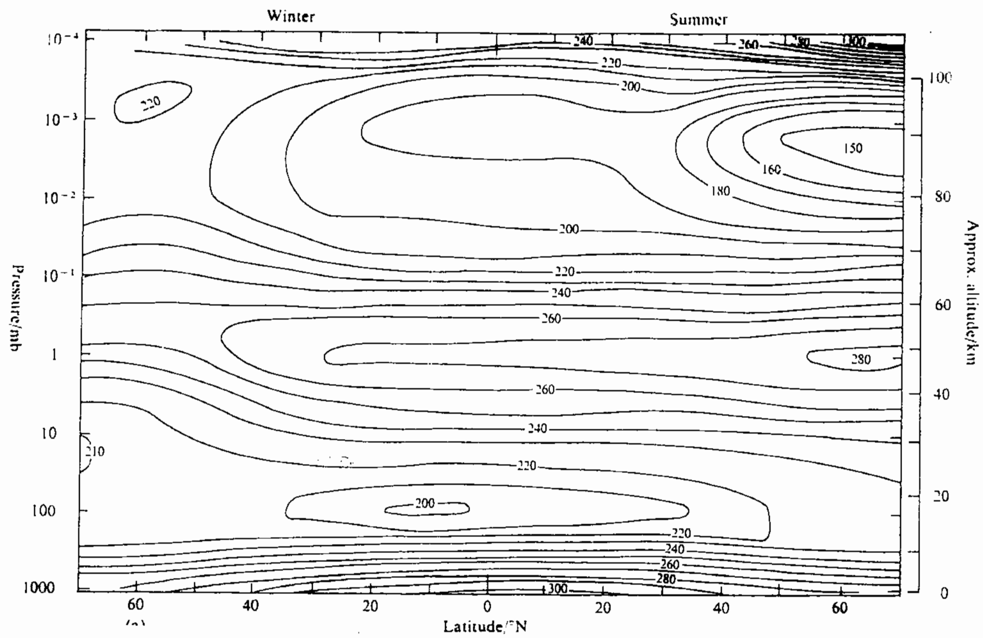

Figure 3.5: The observed, longitudinally averaged temperature distribution in northern summer. [After Houghton, "The Physics of Atmospheres", Cambridge Univ. Press, 1977.]

Since the pressure acting on the parcel changes during displacement, its density will also change, and the two are related to one another *and to temperature* through (3.6). In order to evaluate how density changes we need to know how the temperature changes, which in turn requires knowledge of the parcel's **heat budget** during displacement.

3.5.1 Thermodynamics of dry air

The first law of thermodynamics[2] states that the change in energy, dq, per unit mass of air undergoing temperature and density changes is

$$dq = c_v\, dT + p\, d\alpha \;, \tag{3.12}$$

where c_v is the specific heat at constant volume and $d\alpha$ the change in specific volume (the volume of the unit mass). Since $\alpha = 1/\rho$, $d\alpha = -d\rho/\rho^2$.

[2] Good discussions of elementary atmospheric thermodynamics can be found in Chapter 2 of Wallace & Hobbs, *Atmospheric Science: an Introductory Survey*, (Academic Press, 1977) and Fleagle & Businger, *An Introduction to Atmospheric Physics*, (Academic Press, 1980).

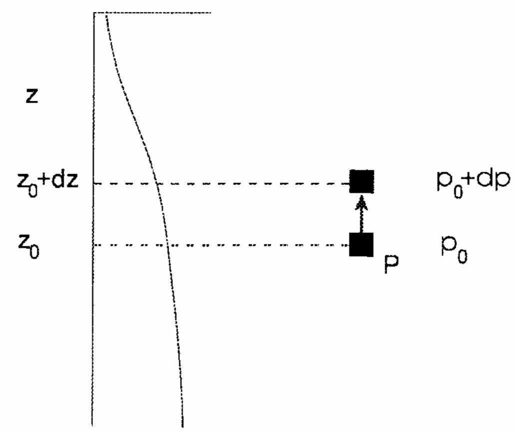

Figure 3.6: Vertical displacement of a compressible air parcel.

Therefore

$$p\,d\alpha = p\ d(\frac{1}{\rho}) = d(\frac{p}{\rho}) - \frac{1}{\rho}dp = R\,dT - \frac{1}{\rho}dp\ ,$$

where we have used the ideal gas law (3.6). Therefore (3.12) can be written

$$dq = c_p\,dT - \frac{1}{\rho}dp\ ,$$

where $c_p = c_v + R$ is the specific heat at constant pressure. To convert this into an equation for the change in heat content *per unit volume*, dQ, we just multiply by ρ to give

$$dQ = \rho c_p dT - dp\ . \tag{3.13}$$

Hence, we can now write the first law in time derivative form (its customary form for meteorological application):

$$\frac{dT}{dt} - \frac{1}{\rho c_p}\frac{dp}{dt} = \frac{J}{\rho c_p}\ , \tag{3.14}$$

where $J = dQ/dt$ is the so-called **diabatic heating rate** per unit volume.

Consider now the quatity

$$\theta = T\left(\frac{p_0}{p}\right)\kappa\ , \tag{3.15}$$

where p_0 is a constant (conventionally taken to be 100 kPa = 1000 mb) and $\kappa = R/c_p = 2/7$ for air. Then

$$\begin{aligned} d\theta &= dT\left(\frac{p_0}{p}\right)\kappa - \kappa T\frac{dp}{p}\left(\frac{p_0}{p}\right)\kappa \\ &= dT\left(\frac{p_0}{p}\right)\kappa - \frac{dp}{\rho c_p}\left(\frac{p_0}{p}\right)\kappa\,, \end{aligned}$$

where we have used (3.6) to show $\kappa T/p = RT/pc_p = 1/\rho c_p$. Therefore (3.14) can be written

$$\frac{d\theta}{dt} = \frac{J}{\rho c_p}\left(\frac{p}{p_0}\right)\kappa\,. \tag{3.16}$$

Eq. (3.16) has the obvious advantage of being more concise than (3.14), but its great power—and the usefulness of the quantity θ, which is known as **potential temperature**—becomes clearest under circumstances in which the diabatic heating J can be neglected. The most important heating (or cooling) processes are:

(i) latent heating or cooling associated with condensation or evaporation of water. This is a very important process, which we will discuss in detail later.

(ii) radiation. On time scales of several days or longer, this is an important process, but is usually weak on shorter time scales.

(iii) conduction. This process is only important **very** close to the surface.

For dry motions, on sufficiently small time scales, and outside the boundary layer, it is usually valid to neglect the diabatic heating, in which case the motions are **adiabatic** and (3.16) becomes simply

$$\frac{d\theta}{dt} = 0\,. \tag{3.17}$$

Potential temperature is thus **conserved** under adiabatic conditions[3]. Unlike temperature, the potential temperature does not change as an air parcel moves adiabatically to higher or lower pressure. Note that at $p = p_0$, $\theta =$

[3] θ is actually a measure of the **specific entropy** of air (which in fact is $c_p \ln\theta$, to within an arbitrary constant), which does not change under adiabatic processes.

T: so, if a parcel at some location in the atmosphere has temperature T and potential temperature θ then, if $p \neq p_0$, θ and T will be different. If we move the parcel adiabatically to the standard pressure, it will still have potential temperature θ, but its temperature will now be $T = \theta$. Therefore, the physical meaning of θ is:

1. The potential temperature of an air parcel is the temperature it would have if moved adiabatically to standard pressure (1000 mbar).

3.5.2 Static stability

Now let's return to the vertically displaced air parcel of Fig. 3.6. If we assume the displacement is rapid (hours or less) and that there is no moisture condensation within the parcel, then we can assume the displacement to be adiabatic, so that $d\theta = 0$ as the parcel is displaced. Now, the parcel leaves height z with initial density $\rho_i = \rho_e(z) = p(z)/RT_e(z),$. where T_e and ρ_e are the environmental temperature and density. At the final height, $z + dz$, the parcel has density

$$\rho_f = \frac{p(z+dz)}{RT_f}$$

and the environmental density is

$$\rho_e(z+dz) = \frac{p(z+dz)}{RT_e(z+dz)} \; .$$

Now the parcel will be **buoyant**—and therefore the displacement will continue to grow—if $\rho_f < \rho_e(z+dz)$, *i.e.*, if $T_f > T_e(z+dz)$. If $T_f < T_e(z+dz)$, however, the parcel will be **negatively buoyant** and will return toward its original position: the environment will then be **statically stable** with respect to displacement. Now, since $d\theta = 0$ for the parcel, its temperature will change according to

$$dT = \frac{dp}{\rho c_p} = -\frac{g}{c_p} dz \tag{3.18}$$

where we have used hydorstatic balance (3.4). Therefore its final temperature is

$$T_f = T_i - \frac{g}{c_p} dz = T_e(z) - \frac{g}{c_p} dz \; ,$$

(assuming it left with environmental temperature). But the environmental temperature at this location is

$$T_e(z+dz) = T_e(z) + \frac{dT_e}{dz}dz \; ;$$

therefore the environment will be

$$\begin{array}{lll} \text{unstable} & if \; \frac{dT_e}{dz} < -\frac{g}{c_p} & \left(\frac{d\theta_e}{dz} < 0\right) \\ \text{stable} & if \; \frac{dT_e}{dz} > -\frac{g}{c_p} & \left(\frac{d\theta_e}{dz} > 0\right) \end{array} . \tag{3.19}$$

The critical value of temperature gradient

$$\frac{dT_e}{dz} = -\Gamma_{ad} = -\frac{g}{c_p} \tag{3.20}$$

is known as the **adiabatic lapse rate.** c_p for air has a value of 1004 J $K^{-1}kg^{-1}$, so $\Gamma_{ad} = 9.8$ K km^{-1}. Usually (though not always), the actual lapse rate of temperature is less than this (typically 6-7 K km^{-1}) so the atmosphere is usually stable to dry displacements of this kind.

3.6 Internal waves in the atmosphere

3.6.1 The buoyancy frequency in a compressible atmosphere

A statically stable atmosphere, like a stably stratifed ocean, will support internal gravity waves. In fact, atmospheric internal waves are almost identical to those in the ocean—satisfying the same dispersion relation (3.2), for example[4]—but there is one major modification to be made. The buoyancy frequency for incompressible waves is proportional to vertical density gradient; in the atmosphere, as we have seen, it is not this that determines buoyancy, but the gradient of potential temperature. So, for the atmosphere, the buoyancy frequency [cf., eq. (3.3)] is defined by

$$N^2 = \frac{g}{\theta}\frac{d\theta}{dz} = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_{ad}\right) . \tag{3.21}$$

[4]There are some minor terms to be added to (3.2) in general, but in practice (3.2) is a good approximation.

Typically, in the troposphere, $N \simeq 0.01\ \mathrm{s}^{-1}$, corresponding to a period for vertical displacements of about 10 minutes (remember that this is a lower limit of the period of internal waves in general).

Like the ocean, the atmosphere is rich in internal waves (they can often be seen in clouds) though under most circumstances, their amplitudes are not very large in the lower atmosphere. One situation in which they are commonly large is when air flows over mountains—we shall look at such waves below.

Because of one other important effect of compressibility, these waves assume much greater importance in the upper atmosphere (especially in the mesosphere, above 50km altitude). As we have seen, such waves can propagate vertically as well as horizontally; as they do, they encounter reduced environmental density. In order to conserve their energy (or something like it), they must increase their amplitude (Fig. 3.7) as they propagate to higher altitudes, rather like when ocean waves run up toward a beach into shallower water. The amplitude grows as something like $\rho^{-1/2}$. As a result, wave amplitudes are much larger in the upper atmosphere than in the lower atmosphere, even though it is in the latter that most of them originate.

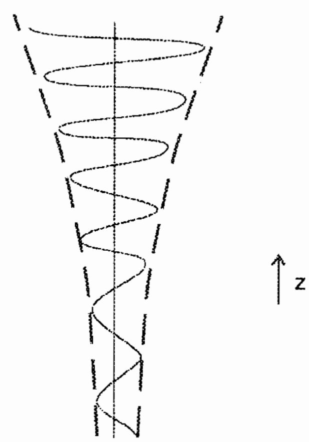

Figure 3.7: Schematic of vertically propagating internal waves.

3.6.2 Mountain waves

Air flowing over mountains produces a stationary wave train, just as a rock in a river produces a train of surface waves. In the former case, for mountains less than about 100km in width (we shall discuss large mountain ranges later) the wave train is comprised of internal waves which in this case are known as **lee waves**. A typical situation is shown in Fig. 3.8.

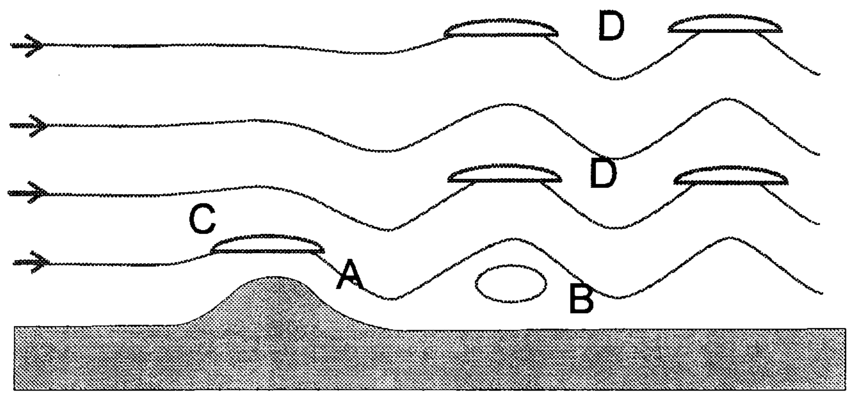

Figure 3.8: Schematic of air flow over a mountain range.

There are several noteworthy features of this flow:

(i) Like the rock-in-the-river problem, in situations where a wave train is produced, it exists downstream of the mountain, and for the same reasons. The wave train is stationary relative to the mountain. Consider the two-dimensional case with y-wavenumber $l = 0$. If the oncoming wind (which we assume to be uniform) is U, then *relative to the flow*, the mountain, and the wave train, have speed $-U$, whence, from (3.2),

$$\frac{\omega}{k} = -U = -\frac{N}{\sqrt{k^2 + m^2}}$$

(the minus sign has been chosen because the propagation is to the left). The x-component of group velocity relative to the flow is

$$c_{gx} = \frac{\partial \omega}{\partial k} = -\frac{Nm^2}{(k^2 + m^2)^{\frac{3}{2}}} = -U\,\frac{m^2}{k^2 + m^2} \geq -U$$

and therefore the group velocity relative to the mountain, $c_{gx} + U \geq 0$: there are no upstream effects[5].

(ii) Immediately in the lee of the mountain, (A on Fig. 3.8) there may be strong, warm, downslope winds. The air is warm because it has come from above, and (since the stratification is stable) $d\theta/dz > 0$, so air from aloft is warmer than surface air if the former is brought down to the surface. The air may also be warmed by latent heating associated with condensation in the upslope flow (see point (v), below).

(iii) Further downstream, there may be strong surface winds where (B on Fig. 3.8) the streamlines concentrate near the surface; these winds may occasionally be extremely strong, but may exist only in a narrow band parallel to the mountain range.

(iv) In regions above point A and above and just upstream of B, there is downward flow. Occasionally, this flow may be manifested as strong downdrafts that can be hazardous to aircraft operating out of or into airports downwind of large mountains.

(v) As the air is elevated over the mountain, condensation may occur, and **orographic clouds** are common (C on Fig. 3.8).

(vi) Clouds also frequently form at one or more levels in the peaks of the lee wave (D on Fig. 3.8). These lee-wave clouds are often seen with banded structure downstream of long ranges, but may also occur with less organization downstream of isolated mountains.

(vii) The lee waves propagate vertically, and so the form drag on the mountain may be communicated by the waves' momentum transport to high levels in the atmosphere. This process is significant enough to be included as an explicit parameterization in numerical weather prediction models, and, at very high levels, also has a dramatic effect on the circulation of the mesosphere.

Finally, we should note that our discussion of internal gravity waves has (for simplicity) been confined to waves on uniform background states (constant N and U). In fact, the most dramatic mountain waves are found where

[5]There may be upstream effects for small U, when no wave train is produced and the flow cannot creep over the mountain, and when nonlinear effects we have not considered may be important.

N and/or U are very nonuniform, in which case **wave trapping**, and consequent amplification, may occur.

3.7 Further reading

Internal gravity waves are covered to some extent in many texts of geophysical fluid dynamics; a detailed but thorough treatment is given in Chapter 6 of A.E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, 1982.

3.8 Appendix to Ch. 3: Theory of internal gravity waves

3.8.1 Stable density stratification in an incompressible fluid

Consider the situation depicted in Fig. 3.9. The water is assumed to be

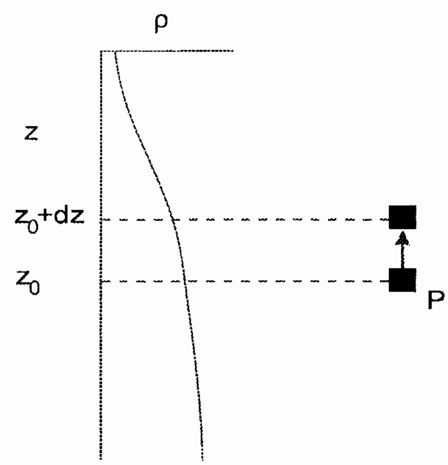

Figure 3.9: Displacement of a water parcel P in stable stratification.

incompressible and to have density varying with depth only, $\rho = \rho(z)$. (N.B. Incompressibility means that density does not change in response to pressure variations; but it does depend on temperature and salinity, $\rho = \rho(T, S)$, so is not spatially constant.) A water parcel P, initially located at $z = z_0$, is displaced upward to $z = z_0 + dz$. The parcel initially had the same density as its environment, $\rho_P = \rho(z_0)$. Now, if we make the reasonable assumptions that there are no sources or sinks of salt within the parcel, and that it moves quickly enough to do so **adiabatically** (without loss or gain of heat), then it preserves its T and S, and thus its density. So, after displacement, its density is still $\rho_P = \rho(z_0)$.

Now, its environment at its new location has density

$$\rho_e = \rho(z_0 + dz) \simeq \rho(z_0) + dz\frac{d\rho}{dz}(z_0) \ ,$$

for small displacement dz. Therefore the parcel will feel a **buoyancy** force causing it to rise further, or to return toward its initial location, depending on whether ρ_e is greater than or less than ρ_P. We will defer discussion of the first possibility—the **unstable** case—until later; we now concentrate on the case of **stable stratification**, *viz.*,

$$\frac{d\rho}{dz} > 0 \ , \tag{3.22}$$

for which the buoyancy is negative and the associated restoring force tends to make the parcel motion oscillatory about its location of neutral buoyancy.

3.8.2 Small amplitude motions in an incompressible fluid with continuous stratification

So, we shall consider inviscid, adiabatic motions in an infinite, two-dimensional $(x-z)$ fluid, with density $\widetilde{\rho}(x,z,t)$. As in Chapter 2, the equations of motion are

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\widetilde{\rho}}\frac{\partial \widetilde{p}}{\partial x} \ , \\ \frac{dw}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\widetilde{\rho}}\frac{\partial \widetilde{p}}{\partial z} - g \ ,\end{aligned}$$

where $\widetilde{p}$ is pressure. We note here that density variations in the ocean are small, and so we can write $\widetilde{\rho} = \rho_{00} + \rho$, where ρ is a small deviation from the constant reference density ρ_{00}. To be consistent, we also have to allow a reference pressure $p_{00}(z)$, in hydrostatic balance with ρ_{00}, such that

$$\frac{dp_{00}}{dz} = -g\rho_{00} \ ,$$

and so we write $\widetilde{p} = p_{00} + p$. Then, relying on the smallness of ρ, we write

$$-\frac{1}{\widetilde{\rho}}\frac{\partial \widetilde{p}}{\partial x} \simeq -\frac{1}{\rho_{00}}\frac{\partial p}{\partial x} + \left\{\frac{\rho}{\rho_{00}^2}\frac{\partial p}{\partial x}\right\},$$

$$
\begin{aligned}
-\frac{1}{\widetilde{\rho}}\frac{\partial\widetilde{p}}{\partial z} &\simeq -\frac{1}{(\rho_{00}+\rho)}\frac{\partial\left(p_{00}+p\right)}{\partial z} \\
&\simeq -\frac{1}{\rho_{00}}\frac{\partial\left(p_{00}+p\right)}{\partial z}+\frac{\rho}{\rho_{00}^{2}}\frac{\partial\left(p_{00}+p\right)}{\partial z} \\
&\simeq g-\frac{1}{\rho_{00}}\frac{\partial p}{\partial z}+g\frac{\rho}{\rho_{00}}+\left\{\frac{\rho}{\rho_{00}^{2}}\frac{\partial p}{\partial z}\right\} .
\end{aligned}
$$

Note that the terms in curly brackets are quadratic in departures from the reference state, so we neglect them, in which case the eqs. of motion become

$$
\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_{00}}\frac{\partial p}{\partial x} , \\
\frac{dw}{dt} &= \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_{00}}\frac{\partial p}{\partial z}+g\frac{\rho}{\rho_{00}} .
\end{aligned}
\tag{3.23}
$$

We also have our incompressible continuity eq.

$$
\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 . \tag{3.24}
$$

To close the problem we need an equation for density. On the basis of our assumption that the motions are adiabatic, and that there are no sources or sinks of salt, it follows that parcels must conserve their density as they move around, *i.e.*,

$$
\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+w\frac{\partial\rho}{\partial z}=0 . \tag{3.25}
$$

Now, we consider a steady, motionless basic state, in which $\rho=\rho_0(z)$ is a linear function of z (for simplicity), such that $d\rho_0/dz=\Lambda$. The second of (3.23) tells us that the basic state pressure field $p_0(z)$ must be in hydrostatic balance with this density field:

$$
\frac{dp_0}{dz}=-g\rho_0(z). \tag{3.26}
$$

We now consider small amplitude perturbations to this state, such that

$$
\begin{aligned}
u &= u'(x,z,t) , \\
w &= w'(x,z,t) , \\
p &= p_0(z)+p'(x,z,t) , \\
\rho &= \rho_0(z)+\rho'(x,z,t) .
\end{aligned}
$$

Since the perturbations are small, we may neglect nonlinear terms like $u'\frac{\partial u'}{\partial x}$ and $w'\frac{\partial \rho'}{\partial z}$; therefore, our linearized perturbation equations become, from (3.23), (3.24), and (3.25),

$$\begin{aligned} \frac{\partial u'}{\partial t} + \frac{1}{\rho_{00}}\frac{\partial p'}{\partial x} &= 0 \\ \frac{\partial w'}{\partial t} + \frac{1}{\rho_{00}}\frac{\partial p'}{\partial z} &= -g\frac{\rho'}{\rho_{00}} \\ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} &= 0 \\ \frac{\partial \rho'}{\partial t} - w'\Lambda &= 0\,. \end{aligned} \tag{3.27}$$

With some juggling[6], these can be reduced to a single equation for p':

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2}\right) + N^2\frac{\partial^2 p'}{\partial x^2} = 0\,. \tag{3.28}$$

The quantity N in (3.28) has units of t^{-1}, and is defined by

$$N^2 = \frac{g\Lambda}{\rho_{00}} = \left(\frac{g}{\rho}\frac{d\rho}{dz}\right)_0\;; \tag{3.29}$$

we will see its significance in a moment.

Clearly, (3.28) supports wavelike solutions of the form

$$p'(x,z,t) = Re\;P\;e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = Re\;P\;e^{i(kx+mz-\omega t)}\;, \tag{3.30}$$

where $\mathbf{k}$ is wavenumber and (k,m) its components in the (x,z) directions, provided

$$\omega = \pm\frac{Nk}{\sqrt{k^2+m^2}}\,. \tag{3.31}$$

Eq. (3.31) is our dispersion relation for internal gravity waves. It tells us that the wave frequency is independent of the *magnitude* of wavenumber, only on its direction; specifically, that

$$\omega = \pm N\sin\vartheta \tag{3.32}$$

[6]Take $\partial/\partial x$ of the 1st of (3.27) plus $\partial/\partial z$ of the 2nd, and use the 3rd to give $g\partial\rho'/\partial z = \partial(\partial^2 p'/\partial x^2 + \partial^2 p'/\partial z^2)/\partial t$; substitute this and the 1st equation into $\partial^2/\partial z\partial t$ of the 4th equation.

where $\vartheta = \arctan(k/m)$ is the angle the wavenumber vector makes with the vertical. For waves with wavenumber pointing horizontally (vertical wave crests), $\omega = \pm N$. So the quantity N defined in (3.29)—which is thus known as the **buoyancy frequency** or the **Brunt-Väisälä frequency**—gives the frequency of such waves; in general (when ϑ is not $\pi/2$) it provides the scale for frequency, although it should be noted that in both ocean and atmosphere, very slow waves with $\omega \ll N$ (so $\vartheta \ll 1$) are common. N is the upper limit of frequency for **propagating** waves, for which both components of wavenumber are real (if $\omega < N$, say because of external forcing at frequency ω, at least one of k and m must be imaginary, and the disturbance will be **evanescent** in at least one direction).

Note from the 3rd of eqs. (3.27), together with (3.30) that

$$ku' + mw' = \mathbf{k} \cdot \mathbf{u}' = 0\ ;$$

the motions are at right angles to the wavenumber, and thus along the phase lines: the wave motion is **transverse**. This is illustrated in Fig. 3.10. When $\vartheta = \pi/2$, the phase lines and motions are aligned vertically—so the

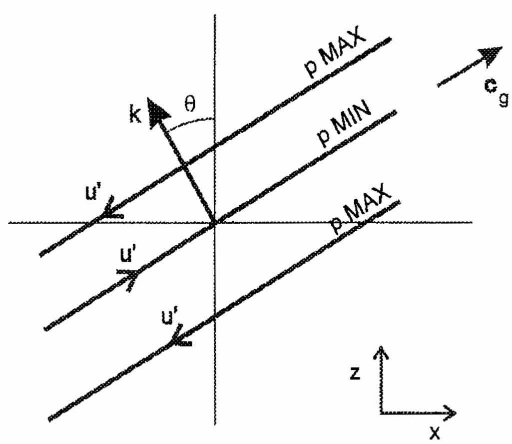

Figure 3.10: Phase lines and motions within a plane internal gravity wave.

oscillation of fluid parcels is just as we discussed at the beginning of this chapter; thus N is the frequency of vertically-displaced parcels. For other values of ϑ, the component of restoring force along the angle of displacement is what matters—hence (3.32).