12.510 Introduction to Seismology
Spring 2008

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Feb. 29th 2008:

We have introduced the equations:

$$\nabla^2\phi(\mathrm{x},t) = \frac{1}{\alpha^2}\frac{\partial^2\phi(\mathrm{x},t)}{\partial t^2} \quad \text{and} \quad \nabla^2\Psi(\mathrm{x},t) = \frac{1}{\beta^2}\frac{\partial^2\Psi(\mathrm{x},t)}{\partial t^2}$$

With $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$ **and** $\beta = (\mu/\rho)^{1/2}$

These equations can be solved using 3 different methods:

1. **D'Alembert's solution (most 'physical' approach)**
2. **Separation of variables**
3. **Fourier Transforms (mathematically, the most powerful method)**

There is a whole class of theoretical development in applied maths that uses Fourier Integral Operators (FIOs)

1.D'Alembert's Solution:

Take as an example, the wave equation:

$$\ddot{\phi} = \alpha^2\nabla^2\phi$$

And the function: $\phi(x,t) = f(x-ct) + g(x+ct)$ **(45)**

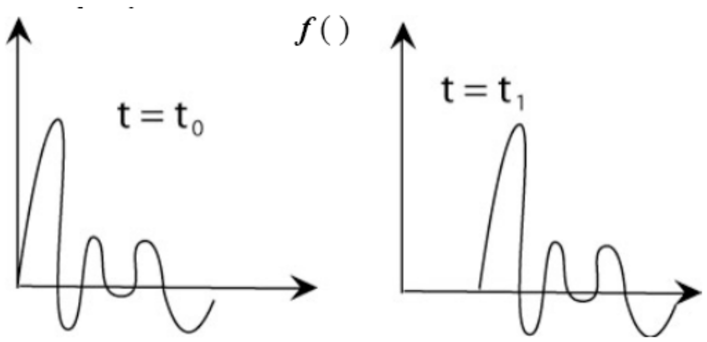
The first term: f(x – ct) represents propogation in the positive x-direction

The second term: g(x+ct) represents propogation in the negative x-direction

c = the wave speed or phase speed/velocity

Consider the profile of the wave at a time t_0 and at some later time t_1

Figure 4:



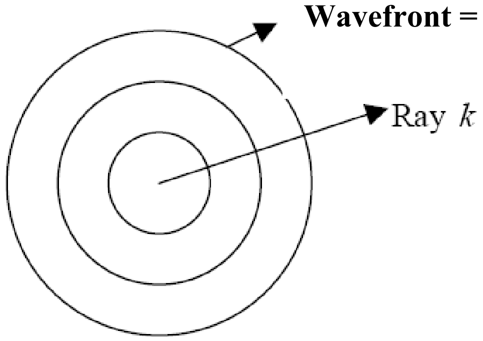
Example of a plane wave propagation in positive x-axis; function $f(\,)$ remains same if argument (the phase!) remains same; this happens if x increases when t increases → motion in positive x axis.

(x-ct) is known as the phase of the wave.

The phase speed is given by: $c - \frac{x_1 - x_0}{t_1 - t_0}$ **(46)**

<u>Figure 5: Diagram to illustrate the concept of wavefronts:</u>

Wavefront = surface connecting points of equal phase



A wavefront is a line in 2d (or surface in 3d) connecting points of equal phase.

In reality, the wavefronts are circular, but locally they behave as a plane wave.

All points along the wave-front have the same travel-time from the origin.

The relationship between the wavenumber (k) **and angular frequency** (ω) **is given by:**

$k = \frac{\omega}{c}$ **(47)**

The relationship between wavenumber (k) **and wavelength** (λ) **is given by:** $k = \frac{2\pi}{\lambda}$ **(48)**

So we can re-write the phase in terms of wavenumber: $\left(\frac{x}{c} - t\right) \rightarrow (kx - \omega t)$ **(in one dimension)**

In 3 dimensions, this becomes: $(k_x x + k_y y + k_z z - \omega t)$ **or** $(k.x - \omega t)$

The harmonic function is a solution to the wave equation:

$$\phi = \cos\left(kx - \omega t\right) + i\sin\left(kx - \omega t\right) = \exp i\left(kx - \omega t\right) \quad (49)$$

Where we have used the identity: $\exp(i\alpha) = \cos(\alpha) + i\sin(\alpha)$ **(50)**

2. Separation of variables:

Using the method of separation of variables, we trial a solution of the wave equation of the form:

$$\phi(x,y,z,t) = X(x)Y(y)Z(z)T(t) \quad (51)$$

Substituting this into the equation: $\ddot{\phi} = c^2\nabla^2\phi$ **gives:**

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0 \quad (52)$$

To satisfy this equation, each term must be equal to a constant and the constants must sum to zero. We choose the constants: $-k_{x'}^2 - k_{y'}^2 - k_{z'}^2\left(\frac{\omega^2}{c^2}\right)$ **respectively**

$$\frac{d^2X}{dx^2} + {k_x}^2X = 0 \longrightarrow e^{\pm ik_xX}$$
$$\frac{d^2Y}{dy^2} + {k_y}^2Y = 0 \longrightarrow e^{\pm ik_yY}$$
$$\frac{d^2Z}{dz^2} + {k_z}^2Z = 0 \longrightarrow e^{\pm ik_zZ}$$
$$\frac{d^2T}{dt^2} + \omega^2T = 0 \longrightarrow e^{\pm i\omega T} \quad (53)$$

Applying the condition that these constants must sum to zero gives us the dispersion relation:

$$k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega}{c}\right)^2 = 0 \text{ dispersion relationship.} \quad (54)$$

Substituting the solutions for X,Y,Z,T back into the original trial solution:

$$\phi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

Gives our final solution of the form: $\phi = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ **(55)**

Note: Wavenumbers and the wavevector:

We have already defined:

$k = \frac{2\pi}{\lambda}$

The wavevector is: $k = \left(k_x, k_y, k_z\right)$ **and gives the "direction" of the wave.**

The length of the wavevector is the wavenumber: $|k| = \frac{\omega}{c} = k$ **(56)**

Our solution is effectively harmonic functions that propogate in the direction of k.

The full solution is a superposition of plane waves:

The displacement is relatec $\phi(\mathbf{k}, \mathbf{x}) = \sum_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ **(57)**

$u = \nabla\phi$ **(58)**

$u(x,t) = \nabla\phi(x,t)$ **(59)**

So $u(x,t) = (0, 0, ik_z) A \exp\{i(k_z z - \omega t)\}$ **(60)**

The displacement vector has harmonic wave character and propogates in the z direction

The imaginary part of the diplacement vector is associated with the amplitude of the wave

The propogating part is found by taking the real part of this displacement vector.

$u(x,t) = (0, 0, ik_z) A \exp\{i(k_z z - \omega t)\} = ik_x\left[\cos(k_z z - \omega t) + i\sin(k_z - \omega t)\right]$ **(61)**

The real part of this is: $\operatorname{Re}\left[u(x,t)\right] = -k_z \sin(k_z - \omega t)$ **(62)**

Note: The Helmholtz equation:

If we consider solutions to the wave equation $\ddot{\phi} = \alpha^2\nabla^2\phi$ **in 1d, of the form** $\phi = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$

We can differentiate with respect to time get:

$\frac{d^2\phi}{dt^2} = -\omega^2\phi$ **(63) Substituting into the wave equation gives:**

$\nabla^2\phi + \frac{\omega^2}{\alpha^2}\phi = 0$ **(64) and** $k = \frac{\omega}{\alpha}$ **so, we have:**

$\nabla^2\phi + k^2\phi = 0$ **(The Helmholtz equation) (65)**

A lot of imaging is done in the frequency domain by finding solutions to the helmholtz equation.

3. Fourier Transforms:

Fourier transforms allow us to understand the relationship between the space-time (x,t) and wavenumber-frequency (k,ω) domains.

In one dimension, the forward and reverse fourier transforms between the space-frequency and space-time domains are given by:

$$\Phi(\mathbf{x},\omega) = \int_{-\infty}^{\infty} \phi(\mathbf{x},t) e^{i\omega t} dt \longleftrightarrow \phi(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\mathbf{x},\omega) e^{-i\omega t} d\omega \quad (66)$$

(Note: in seismology, we normally take the exponential as having a positive sign when we are transforming into the space-frequency domain, however this is simply a convention)

Similarly, we can use fourier transforms to convert between the space-time and wavenumber-time domains. In 3d the fourier transforms between the space-time and wavenumber-time domains are:

$$\Phi(\mathbf{k},t) = \int_{\mathbf{V}} \phi(\mathbf{x},t) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r \longleftrightarrow \phi(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int_{\mathbf{k}} \Phi(\mathbf{k},t) e^{-i\mathbf{k}\cdot\mathbf{x}} dk_x dk_y dk_z \quad (67)$$

Combining these gives the double-fourier transform:

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\iint_{-\infty}^{\infty} \Phi(k_x,k_y,\omega,z) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} dk_x dk_y d\omega \quad (68)$$

Note that k_z does not appear in this equation. This is because, k_x, k_y, k_z are related via the dispersion relation.

$$k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega}{c}\right)^2 = 0 \text{ dispersion relationship.} \quad (69)$$

Hence, if we have specified the angular frequency, k_x and k_y it follows that k_z has already been determined.

Numerically, this double fourier transform is very difficult to work with.

Note: Synthetic seismograms:

A synthetic seismogram is given by a plane-wave superposition.

Suppose we want to make a synthetic seismogram that looks similar to the wave.

We do not need to integrate over the full range $-\pi < k_x < \pi$ **and** $-\pi < k_y < \pi$, **because this implies we do not know anything about the direction of the wave.**

A synthetic seismogram can be produced by limiting the integration over directions $k_0 \pm dk$ **and frequency** $\omega_0 \pm d\omega$.

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int_{\omega_0 - d\omega}^{\omega_0 + d\omega} \iint_{k_0 - dk}^{k_0 + dk} \Phi(k_x, k_y, \omega, z) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} dk_x dk_y d\omega \quad (70)$$

The integrand $\phi(k_x, k_y, \omega z)$ **is the amplitude or weight.**

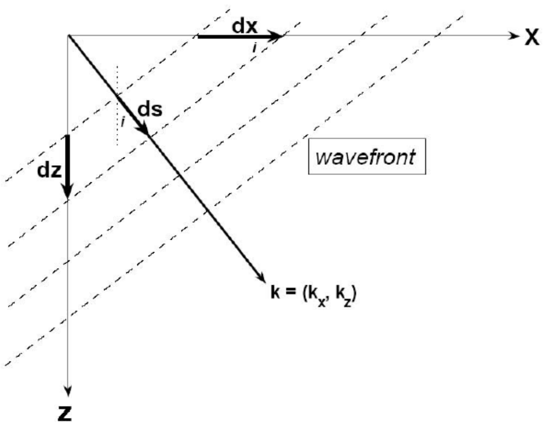
<u>Slowness:</u>

We have see already that the modulus of the wave-vector gives the wavenumber:

$$|k| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{w}{c}$$

In 2d, we have: $$|k| = \sqrt{k_x^2 + k_z^2} = \frac{w}{c} \quad (71)$$

<u>Figure 6:</u>



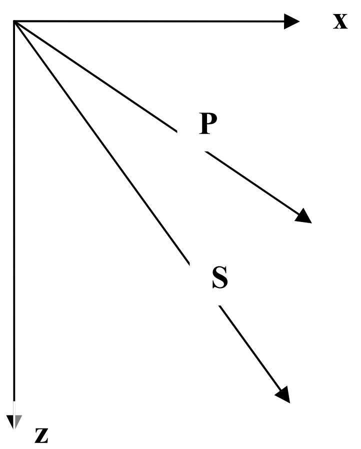
In figure 6, the arrow is used for a ray and the dashed line is used for a wavefront. The wavenumber k indicates the direction of the ray. The angle i is both the 'take-off' angle and the 'angle of incidence'

For a P-wave, the wavenumber is given by: $k_\alpha = \frac{\omega}{\alpha}$ **and for the S-wave, the wavenumber is given by** $k_\beta = \frac{\omega}{\beta}$

Since, in general: $\beta < \alpha$ **it follows directly, that:** $k_\beta = \frac{\omega}{\beta} > k_\alpha = \frac{w}{\alpha}$ **(72)**

Since $k_\alpha = \frac{\omega}{\alpha}$ gives the length of the vector representing the P-wave and $k_\beta = \frac{\omega}{\beta}$ represents the length of the vector representing the S-wave, it follows directly that P-waves 'dive' less steeply into the medium than S-waves.

<u>Figure 7:</u>



The phase 'speed' c, is given by: $c = \frac{ds}{dt}$ **(73) and is a vector in the direction of propogation**

At the surface, we measure: $c_x = \frac{dx}{dt}$ **(74) which is the 'apparent' velocity/speed**

<u>Horizontal slowness:</u>

$$\sin(t) = \frac{ds}{dx} = c\frac{dt}{dx} = c\left(\frac{1}{c_x}\right) = cp \qquad \textbf{(75)}$$

$$\rho = \frac{1}{c_x} = \frac{\sin(t)}{c}$$ **= horizontal slowness = ray parameter (76)**

This follows from Snell's law

<u>Vertical slowness:</u>

We know that: $c_x = (c_x, c_z)$

The vertical slowness is given by: $$\eta \equiv \frac{1}{c_z} = \frac{cos(i)}{c} \qquad \textbf{(77)}$$

Combining the vertical and horizontal slowness:

$$\rho^2 + \eta^2 = \frac{\sin^2 t}{c^2} + \frac{\cos^2 t}{c^2} = \frac{1}{c^2} \qquad \textbf{(78)}$$

Rearranging this gives:

$$\eta = \sqrt{1/c^2 - p^2} \qquad \textbf{(79)}$$

So, the vertical slowness does change with depth, because c is a function of depth. The vertical slowness (η) is zero if $\frac{1}{c^2} = p^2$ (which represents a horizontally propagating wave)

η is imaginary for evanescent waves. (This is important for understanding the behaviour of surface waves).

There is a direct relationship between the wave-vector and the slowness components:

$$k = \frac{\omega}{c}$$

$$k_x = \frac{\omega}{c_x} = \omega p \quad \textbf{(80)}$$

$$k_z = \frac{\omega}{c_z} = \omega\eta \quad \textbf{(81)}$$

$$k = (k_x, k_z) = (\omega p, \omega\eta) = \omega(p, \eta) \quad \textbf{(82)}$$

Notes: Katie Atkinson, Feb 2008

Figures from notes of Patricia M Gregg (Feb 2005) Kang Hyeun Ji (Feb 2005)