

12.520 Lecture Notes 15

Special Cases

In general $\tau_{ij} = \lambda e_{ii} \delta_{ij} + 2\mu e_{ij}$

Plane stress: (e.g., $\tau_{zz} = 0$, $\tau_{xz} = 0$)

$$\underline{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p = -\frac{\tau_{xx} + \tau_{yy}}{3}$$

$$\underline{\tau}^{\text{dev}} = \begin{bmatrix} \frac{2\tau_{xx} - \tau_{yy}}{3} & \tau_{xy} & 0 \\ \tau_{xy} & \frac{2\tau_{yy} - \tau_{xx}}{3} & 0 \\ 0 & 0 & \frac{-\tau_{xx} - \tau_{yy}}{3} \end{bmatrix}$$

$$\tau_{zz}^{\text{dev}} = -\frac{\tau_{xx} + \tau_{yy}}{3}$$

Solving for strains:

$$e_{xx} = \frac{1}{E}(\tau_{xx} - \nu\tau_{yy})$$

$$e_{yy} = \frac{1}{E}(\tau_{yy} - \nu\tau_{xx})$$

$$e_{zz} = -\frac{\nu}{E}(\tau_{xx} + \tau_{yy})$$

$$e_{xy} = \frac{\tau_{xy}}{2\mu}$$

Plane strain: (e.g., $e_{zz} = e_{xz} = 0 = \tau_{zz}^{\text{dev}} \Rightarrow \tau_{xx}^{\text{dev}} = -\tau_{yy}^{\text{dev}}$)

$$\underline{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \frac{\tau_{xx} + \tau_{yy}}{2} \end{bmatrix}$$

$$p = -\tau_{zz}$$

$$\underline{\tau}^{\text{dev}} = \begin{bmatrix} \frac{\tau_{xx} - \tau_{yy}}{2} & \tau_{xy} & 0 \\ \tau_{xy} & \frac{\tau_{yy} - \tau_{xx}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{xx} = \frac{1+\nu}{E}[\tau_{xx}(1-\nu) - \nu\tau_{yy}]$$

$$e_{yy} = \frac{1+\nu}{E}[\tau_{yy}(1-\nu) - \nu\tau_{xx}]$$

$$e_{xy} = \frac{\tau_{xy}}{2\mu}$$

Note:

| | | |
|------------------------------------|---|---------------------|
| Plate stress | → | Plane strain |
| E | → | $\frac{E}{1-\nu^2}$ |
| ν | → | $\frac{\nu}{1-\nu}$ |
| $\frac{2\lambda\mu}{\lambda+2\mu}$ | ← | λ |

Plane stress: (e.g., $\tau_{33} = 0$)

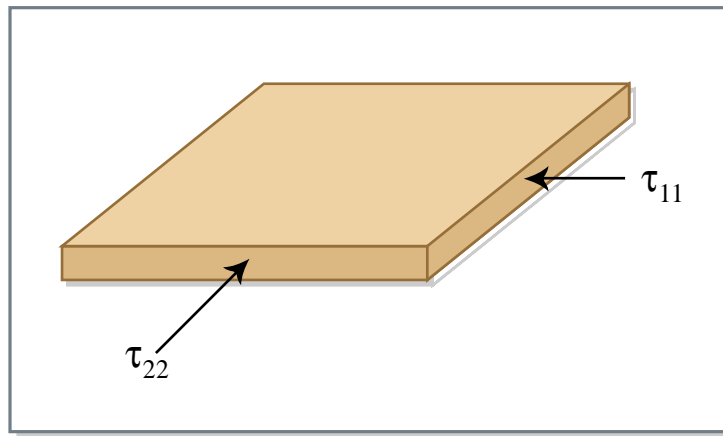


Figure 15.1
Figure by MIT OCW.

$$e_{11} = \frac{1}{E}(\tau_{11} - \nu\tau_{22})$$

$$e_{22} = \frac{1}{E}(\tau_{22} - \nu\tau_{11})$$

$$e_{33} = -\frac{\nu}{E}(\tau_{11} + \tau_{22})$$

Tectonic Stress – Often plane stress is useful (3D → 2D).

Assume mostly in plate stress

Assume

$$\tau_{zz} = -\rho g z \quad (\text{lithostatic})$$

$$\tau_{xx} = -\rho g z + \Delta \tau_{xx}$$

$$\tau_{yy} = -\rho g z + \Delta \tau_{yy}$$

The non-lithostatic (as opposed to deviatoric) stress:

$$\left. \begin{array}{l} \tau'_{zz} = 0 \\ \tau'_{xx} = \Delta \tau_{xx} \\ \tau'_{yy} = \Delta \tau_{yy} \end{array} \right\} \text{plane stress}$$

Assume for simplicity $\Delta \tau_{xx} = \Delta \tau_{yy} = \sigma$

$$e_{11} = e_{22} = \frac{(1-\nu)}{E} \sigma \Rightarrow \text{compression for } \sigma \text{ compression}$$

$$e_{33} = -\frac{2\nu}{E} \sigma \Rightarrow \text{extension for } \sigma \text{ compression}$$

$$\text{dilation } \theta = (2 - 4\nu) \frac{\sigma}{E} = -\frac{\delta\rho}{\rho}$$

Look at the change in lithostatic pressure at a depth $h \rightarrow h + \delta h = h(1 - 2\nu \frac{\sigma}{E})$

$$\rho g h \rightarrow (\rho + \delta\rho) g (h + \delta h)$$

Plane stress, $\sigma_1 = \sigma_2 = \sigma$

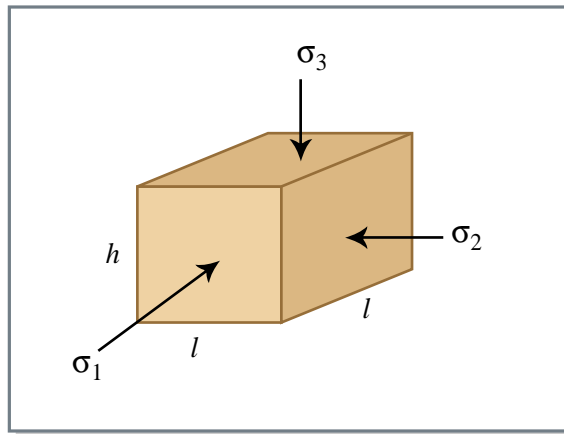


Figure 15.2

Figure by MIT OCW.

$$\varepsilon_1 = \varepsilon_2 = \frac{(1-\nu)}{E} \sigma$$

$$\varepsilon_3 = \frac{-2\nu}{E} \sigma$$

$$\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{(2-4\nu)}{E} \sigma = \frac{2(1-2\nu)}{E} \sigma$$

Now to conserve mass, $\frac{\delta\rho}{\rho} = -\frac{\delta V}{V} = \theta$

New lithostat:

$$\begin{aligned} \delta\tau_{zz} &= -(\rho + \delta\rho)g(h + \delta h) + \rho gh \\ &= (1 + \theta)g\left(1 + \frac{2\nu}{E}\sigma\right)\rho h - \rho gh \\ &= \left(\theta + \frac{2\nu}{E}\sigma\right)\rho gh \end{aligned}$$

$$\frac{\delta\tau_{zz}}{\sigma} = \frac{2(1-\nu)}{E} \rho gh$$

Use

$$\nu = 0.25$$

$$E = 100 \text{ GPa} = 1 \text{ Mbar}$$

$$h = 100 \text{ km}$$

$$\rho = 3 \text{ Mg/m}^3 = 3 \text{ g/cm}^3$$

$$\frac{\delta\tau_{zz}}{\sigma} = \frac{2(1-\nu)}{E} \rho gh ; 5\%$$