5. Initial value problem - Homogeneous medium

Deep water waves ($\omega^2 = gk$). Again method of stationary phase

$$\text{Put} \quad \eta(x,t) = \int_{-\infty}^{+\infty} [C(k)e^{i(kx+\omega t)} + D(k)e^{i(kx-\omega t)}]dk$$

superposition of waves going in opposite directions

We need to specify at t = o

$$\eta(x,o) = \int_{-\infty}^{+\infty} [C(k) + D(k)]e^{ikx}dk$$

and

$$\eta_t(x,o) = \int_{-\infty}^{+\infty} iw[C(k) - D(k)]e^{ikx}dk$$

Assume for simplicity $\eta_t(x,o) = 0$

then C(k) = D(k) and $C(k) + D(k) = 2C(k) = \tilde{\eta}_o$

Then

$$\eta(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{\eta}_o(k)\left[e^{i(kx+\omega t)} + e^{i(kx-\omega t)}\right]dk$$

Now $\omega = (\text{sign } k)(g|k|)^{1/2}$ to ensure that waves at frequency ω propagate in the same direction (even though oppositely) regardless of the sign of k. Separate into left and right-going wave contributions:

$$\eta(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{\eta}_o(k)e^{i\theta^+ t}dk + \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{\eta}_o(k)e^{i\theta^- t}dk$$

where

$$\theta^{\pm} \equiv \frac{kx}{t} \pm (\text{sign } k)(g|k|)^{1/2}$$

Points of stationary phase (giving the major contribution to the integrals) are those where

$$\frac{\partial}{\partial k}\theta^{\pm} = 0$$

Let us consider the half-plane (x>o) for k>o |k| = R

$\theta^{+}(k) = \frac{kx}{t} + (gk)^{1/2}$ left-going wave

$$\frac{\partial\theta^{+}}{\partial k} = \frac{x}{t} + \frac{1}{2}\sqrt{\frac{g}{k}} = 0$$

gives $x = -\frac{t}{2}\sqrt{\frac{g}{k}} < 0$ always for increasing time

No stationary points in θ^{+} for x>o

$\theta^{-} = \frac{kx}{t} - \frac{1}{2}(gk)^{1/2} = 0$ right going wave

$\frac{\partial\theta^{-}}{\partial k}|_{k=k_o} = \frac{x}{t} - \frac{1}{2}(\frac{g}{k})^{1/2} = 0$ at $k = k_o$

$\frac{x}{t} - \frac{1}{2}(\frac{g}{k_o})^{1/2} |= c_g|_{k_o}$ group velocity of packet centered at k_o

$\frac{\partial^2\theta^{-}}{\partial k^2}|_{k_o} = \frac{1}{4}(\frac{g^{1/2}}{k_o^{3/2}}) = \frac{2x^3}{gt^3}$ as $\frac{1}{k_o^{3/2}} = \frac{8x^3}{t^3}\frac{1}{g^{3/2}}$

only right going wave

Then $\eta_{k>o}(x > o,t) \simeq \frac{1}{2}\tilde{\eta}_o(k_o)e^{i\theta^{-}(k_o)t} \int_{-\infty}^{+\infty} e^{\frac{(k-k_o)^2\theta''(k_o)t}{2i}} dk$

As $\int_{-\infty}^{+\infty} e^{-\alpha z^2}dz = (\frac{\pi}{\alpha})^{1/2}$ and $z^2 = (k-k_o)^2$

$$\alpha = \frac{\theta''(k_o)t}{2i}$$

$$\eta_{k>o}(x > o,t) \simeq \frac{1}{2}\tilde{\eta}_o(k_o)e^{i\theta^{-}(k_o)t}\left[\frac{2\pi i}{t\theta''(k_o)}\right]^{1/2}$$

But $\frac{2\pi i}{t\theta''(k_o)} = e^{i\frac{\pi}{2}}\left(\frac{\pi g t^2}{x^3}\right)$

Hence

$$\theta^{-}(k_o) = \left\lfloor k_o \frac{x}{t} - g^{1/2} k_o^{1/2} \right\rfloor t$$

$$= \left\lfloor k_o \frac{g^{1/2}}{2k_o^{1/2}} - g^{1/2} k_o^{1/2} \right\rfloor t = -\frac{1}{2} g^{1/2} k_o^{1/2} t$$

As $k_o^{3/2} = \frac{g^{3/2} t^3}{8x^3} \Rightarrow k_o^{1/2} = \frac{g^{1/2} t}{2x}$

And

$$\theta^{-}(t_o) = -\frac{1}{2} g^{1/2} t \frac{g^{1/2} t}{2x} = -\frac{1}{4} \frac{gt^2}{x}$$

So $$e^{i\theta^{-}(k_o)t} \left[\frac{2\pi i}{t\theta''(k_o)} \right]^{1/2} = e^{-\frac{1}{4} i \frac{gt^2}{x}} e^{i\frac{\pi}{4}} \left[\frac{\pi g t^2}{x^3} \right]^{1/2} = e^{-i\left[\frac{gt^2}{4x} - \frac{\pi}{4}\right]} \left[\frac{\pi g t^2}{x^3} \right]^{1/2}$$

Take the real part

$$\eta_{k>o}(x > o, t) \cong \frac{1}{2} \tilde{\eta}_o(k_o) \left[\frac{\pi g t^2}{x^3} \right]^{1/2} \cos\left[\frac{gt^2}{4x} - \frac{\pi}{4} \right]$$

$$\sim \frac{1}{2} \tilde{\eta}_o(k_o)(\pi g)^{1/2} \left(\frac{t}{x^{3/2}}\right) \cos\left| \frac{gt^2}{4x} - \frac{\pi}{4} \right|$$

$\left[\frac{\pi g t^2}{x^3} \right]^{1/2}$ is the modulating amplitude

Central wavelength $\frac{2\pi}{k_o} = \frac{8\pi x^2}{gt^2}$

x -> increases λ -> increases at fixed x

The final solution is:

$$\eta(x>o,t) \simeq \frac{1}{2}\tilde{\eta}_o(k_o)(\pi g)^{1/2}\left(\frac{t}{x^{3/2}}\right)\cos\left[\frac{gt^2}{4x}-\frac{\pi}{4}\right]$$

modulating amplitude — wave part

Plot η(x,t) as a function of x at fixed t, i.e. a snapshot of the wave:

The central wavelength $\frac{2\pi}{k_o}$ increases and the amplitude decreases with x going away from the initial position.

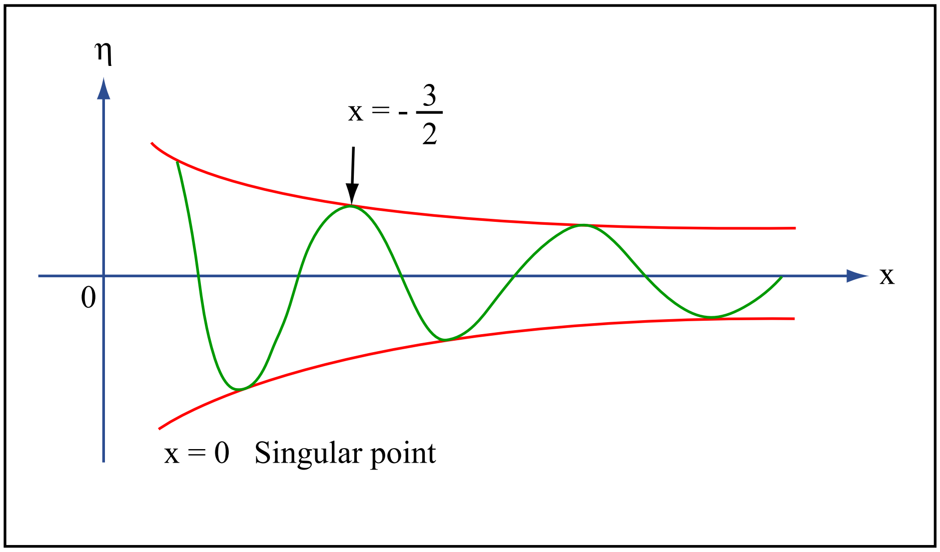
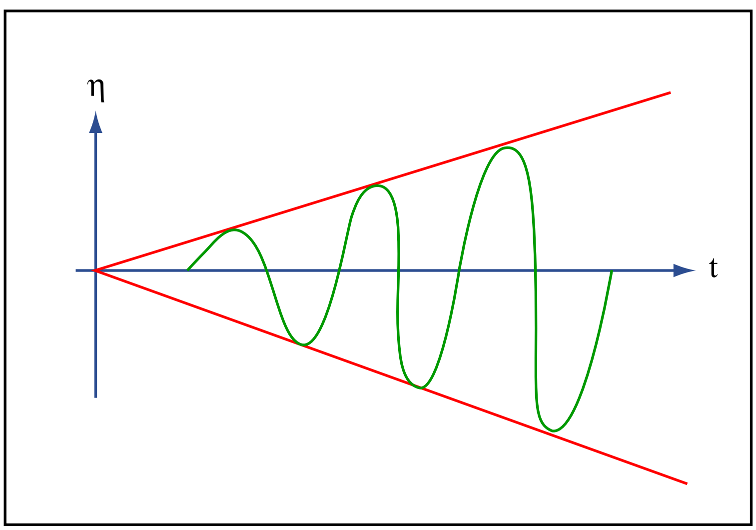


Figure by MIT OpenCourseWare.

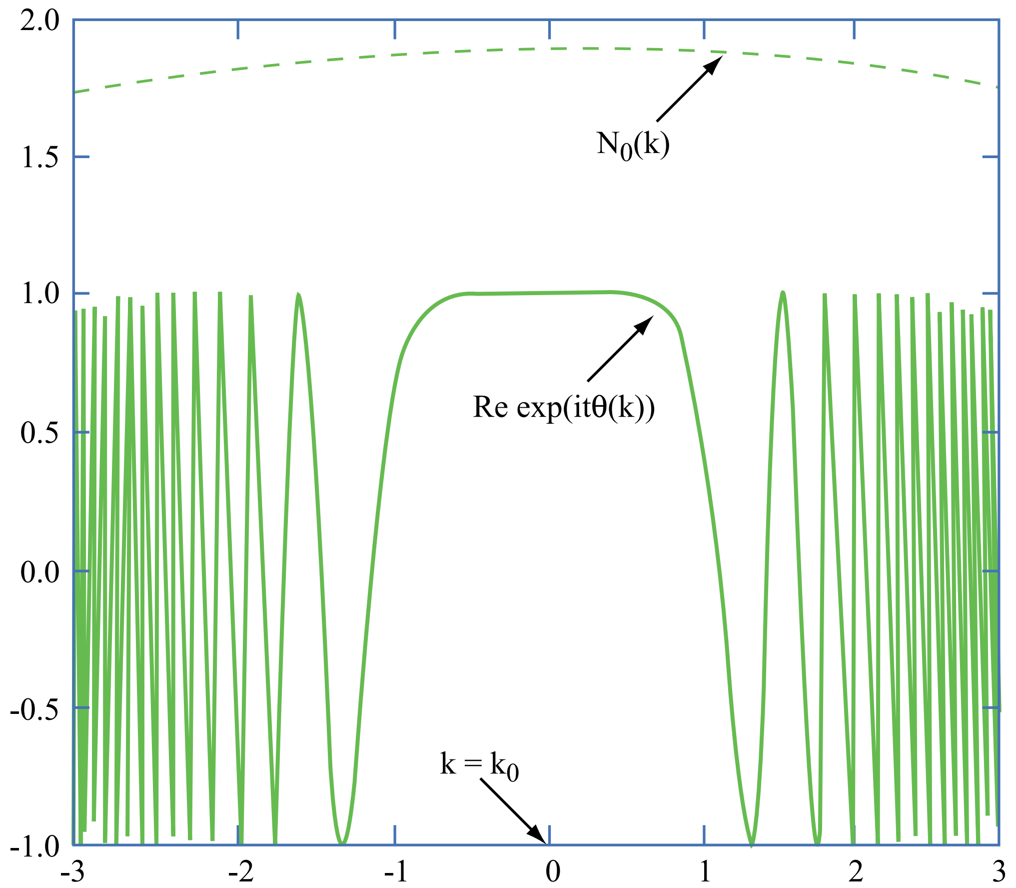
If we put a wavestaff at fixed x and record η(x,t) as a function of time



The wavelength decreases and the amplitude increases linearly

t = o   amplitude = o

Figure by MIT OpenCourseWare.



The behavior of the exponential factor for a large t showing the interval of stationary phase

Figure by MIT OpenCourseWare.

12.802 Wave Motion in the Ocean and the Atmosphere
Spring 2008