Available Potential energy

The available potential energy (Lorenz, 1960, *Tellus*) is defined as the difference between the potential energy of a given density distribution $\rho(x,y,z)$ and the potential energy of the state where all the isentropes are leveled, preserving the amount of mass under them, and arranged in a stably stratified state. Call the resulting density after the rearrangement $\overline{\rho}(z)$. Therefore

$$A = \int d^3\mathbf{x}\; gz[\rho(x,y,z) - \overline{\rho}(z)]$$

For a Boussineq fluid, the density is essentially the same as the entropy, so that this is equivalent (except for a factor of ρ_0) to

$$A = \int d^3\mathbf{x}\; gz[\overline{b}(z) - b(x,y,z)]$$

An isentropic surface is defined by

$$\begin{aligned} b(x,y,z_0 + \xi(x,y,z_0)) &= b_0 \quad (constant) \\ &= \overline{b}(z_0) \end{aligned}$$

where the displacement of the surface is defined to have zero average so that, when leveled, this isentrope will lie at z_0

$$\int dx\, dy\; \xi(x,y,z_0) = 0$$

For small amplitude displacements, we can invert

$$z = z_0 + \xi(x,y,z_0) \quad or \qquad z_0 - z - \xi(x,y,z_0)$$

To get by successive approximation (or iteration)

$$\begin{aligned} z_0 &\simeq z \\ &\simeq z - \xi(x,y,z) \\ &\simeq z - \xi(x,y,z - \xi(x,y,z)) \simeq z - \xi(x,y,z) + \xi(x,y,z)\frac{\partial}{\partial z}\xi(x,y,z) \end{aligned}$$

Therefore

$$\begin{aligned} b(x,y,z) &= \overline{b}(z - \xi(x,y,z) + \xi(x,y,z)\frac{\partial}{\partial z}\xi(x,y,z)) \\ &= \overline{b}(z) - \xi\frac{\partial \overline{b}}{\partial z} + \xi\frac{\partial \xi}{\partial z}\frac{\partial \overline{b}}{\partial z} + \frac{1}{2}\xi^2\frac{\partial^2 \overline{b}}{\partial z^2} \\ &= \overline{b}(z) - \xi\frac{\partial \overline{b}}{\partial z} + \frac{\partial}{\partial z}\left(\frac{1}{2}\xi^2\frac{\partial \overline{b}}{\partial z}\right) \end{aligned}$$

Using this, we find

$$
\begin{aligned}
A &\simeq g \int d^3\mathbf{x}\ \xi z \frac{\partial \bar{b}}{\partial z} - z \frac{\partial}{\partial z} \left(\frac{1}{2} \xi^2 \frac{\partial \bar{b}}{\partial z} \right) \\
&\simeq g \int d^3\mathbf{x}\ \frac{1}{2} \xi^2 \frac{\partial \bar{b}}{\partial z} \\
&\simeq g \int d^3\mathbf{x}\ \frac{1}{2} \left(\frac{\bar{b} - b}{\frac{\partial \bar{b}}{\partial z}} \right)^2 \frac{\partial \bar{b}}{\partial z} \\
&\simeq g \int d^3\mathbf{x}\ \frac{1}{2} \frac{b'^2}{\frac{\partial \bar{b}}{\partial z}} = g \int d^3\mathbf{x}\ \frac{1}{2} \frac{b'^2}{N^2}
\end{aligned}
$$