QG equations

1 — Stratification

We begin with the hydrostatic, Boussinesq equations, with $\mathbf{u}$ being the *horizontal* velocity and w being the vertical velocity

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla P \tag{1}$$

$$0 = -\frac{\partial P}{\partial z} + b$$

$$\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

$$\frac{D}{Dt}b + wN^2 = 0$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + w\frac{\partial}{\partial z}$$

The thermodynamic and hydrostatic equations combine to give

$$\frac{D}{Dt}\frac{\partial P}{\partial z} + wN^2 = 0 \tag{3}$$

The set of equations (1, 2, 3) now completely describe the system.

2 — Geostrophy

We rewrite the momentum equations as

$$\begin{aligned}\mathbf{u} &= \hat{\mathbf{k}} \times \frac{1}{f}\nabla P + \hat{\mathbf{k}} \times \frac{1}{f}\frac{D}{Dt}\mathbf{u} \\ &= \mathbf{u}_g + \hat{\mathbf{k}} \times \frac{1}{f}\frac{D}{Dt}\mathbf{u}\end{aligned}$$

and use the geostrophic velocity to approximate the last term

$$\mathbf{u} \simeq \mathbf{u}_g + \hat{\mathbf{k}} \times \frac{1}{f}\frac{D_g}{Dt}\mathbf{u}_g$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla$$

This uses the smallness of the vertical advection (note #1). The conservation of mass equation gives approximately

$$\nabla \cdot \mathbf{u}_g + \nabla \cdot \hat{\mathbf{k}} \times \frac{1}{f}\frac{D_g}{Dt}\mathbf{u}_g = -\frac{\partial w}{\partial z}$$

From the definition of the geostrophic velocity, we find

$$\nabla \cdot \mathbf{u}_g = -\frac{\beta}{f^2} P_x = -\frac{\beta}{f} v_g = -\frac{1}{f}\frac{D_g}{Dt}\beta y$$

The second term gives

$$-\hat{\mathbf{k}} \cdot \nabla \times \frac{1}{f}\frac{D_g}{Dt}\mathbf{u}_g = -\frac{1}{f}\hat{\mathbf{k}} \cdot \nabla \times \frac{D_g}{Dt}\mathbf{u}_g + O(\beta L/f)$$

which becomes

$$-\frac{1}{f}\frac{D_g}{Dt}\hat{\mathbf{k}} \cdot \nabla \times \mathbf{u}_g = -\frac{1}{f}\frac{D_g}{Dt}\zeta_g$$

(note #2).

Thus the conservation of mass reduces to the QG vorticity equation

$$\frac{D_g}{Dt}(\zeta_g + \beta y) = f\frac{\partial w}{\partial z} \tag{4}$$

The entropy equation simplifies to

$$\frac{D_g}{Dt}\frac{\partial P}{\partial z} + wN^2 = 0$$

since the vertical velocity is weak and the horizontal/temporal variations in N^2 are small (note #3). Eliminating w from these two gives the QGPV equation

$$\frac{D_g}{Dt}q = 0$$

with

$$q = \zeta_g + \beta y + f\frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}P$$

(note #4) with the geostrophic vorticity given by

$$\zeta_g = \frac{1}{f}\nabla^2 P$$

and

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \frac{1}{f}P_x\frac{\partial}{\partial y} - \frac{1}{f}P_y\frac{\partial}{\partial x}$$

Thus we have an inversion relationship for the geopotential height field given the potential vorticity and an advection equation involving also the geopotential. The combination is a single prognostic equation for the geopotential.

We can also write this in terms of the geostrophic streamfunction, $\psi = P/f_0$ as

$$q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}\frac{f_0^2}{N^2(z)}\frac{\partial}{\partial z}\psi$$

and

$$\frac{\partial}{\partial t}q + J(\psi, q) = 0$$

In the atmosphere, the quasi-Boussinesq approximationis made, replacing the mass equation by

$$\nabla \cdot (\overline{\rho}\mathbf{u}) + \frac{\partial}{\partial z}(\overline{\rho}w) = 0$$

with $\overline{\rho}(x)$ representing the rapid decay of density with height associated with the gas compressibility. It gives a virtually identical result except

$$q = \nabla^2\psi + \beta y + \frac{1}{\overline{\rho}(z)}\frac{\partial}{\partial z}\frac{\overline{\rho}(z) f_0^2}{N^2(z)}\frac{\partial}{\partial z}\psi$$

Notes

#1 — Vertical advection

From the entropy equation, using the geostrophic estimate $P \sim fUL$, we estimate

$$w \sim \frac{U}{L}\frac{fUL}{H}\frac{1}{N^2}$$

Therefore the ratio of vertical to horizontal advection is

$$[w/H]/[U/L] = \frac{fUL}{N^2H^2}$$

or

$$[w/H]/[U/L] = \frac{U}{fL}\frac{f^2L^2}{N^2H^2}$$

The last factor is order one or less so that the vertical advection is a factor of Rossby number smaller thatn the horizontal advection. .

#2 — Pulling the curl inside

In Cartesian coordinates

$$\hat{\mathbf{k}} \cdot \nabla \times \frac{D_g}{Dt}\mathbf{u}_g = \frac{D_g}{Dt}\zeta_g + \zeta_g(\frac{\partial}{\partial x}u_g + \frac{\partial}{\partial y}v_g) = \frac{D_g}{Dt}\zeta_g - \frac{\zeta_g}{f}\frac{D_g}{Dt}\beta y$$

The last term is order $\beta L/f$ compared to the first and can be neglected.

#3 — Smallness of entropy perturbations

The magnitude of $\frac{\partial b}{\partial z}$ compared to N^2 is $fUL/H^2N^2 = \frac{U}{fL}\frac{f^2L^2}{N^2H^2}$.

4 — *Finding* $\frac{\partial w}{\partial z}$

From the entropy equation,

$$w = -\frac{D_g}{Dt}\frac{\frac{\partial P}{\partial z}}{N^2}$$

(since N^2 is a function only of z). When we take a z derivative of this we have

$$\frac{\partial w}{\partial z} = -\frac{D_g}{Dt}\left[\frac{\partial}{\partial z}\frac{\frac{\partial P}{\partial z}}{N^2}\right] - \frac{1}{f}J(\frac{\partial P}{\partial z}, \frac{\frac{\partial P}{\partial z}}{N^2})$$

The last term vanishes, so

$$f\frac{\partial w}{\partial z} = -f\frac{D_g}{Dt}\left[\frac{\partial}{\partial p}\frac{1}{N^2}\frac{\partial P}{\partial z}\right]$$

and we can again pull the f inside, dropping $\beta L/f$ terms.