20. The Charney-Stern Theorem

The Eady problem of baroclinic instability described in section 19a was shown to be remarkably similar to the Rayleigh instability of barotropic flow described in Chapter 18. Both problems can be described in terms of phase-locked, counter-propagating Rossby waves. In section 18.2, we presented Rayleigh's and Fjørtoft's theorems for necessary conditions for the instability of phase-locked barotropic Rossby waves. In 1962, Jule Charney and Melvyn Stern published a generalization of these theorems to the case of three-dimensional, quasi-geostrophic flow.

We begin with equation (9.10) for the conservation of pseudo-potential vorticity, which for an inviscid , adiabatic flow may be written

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) [\frac{1}{f_0}\nabla^2\varphi + \beta y + f_0\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right)] = 0. \qquad (20.1)$$

Now consider the case of infinitesimal perturbations to a background zonal flow that varies only with latitude and altitude:

$$\varphi = \overline{\varphi}(y, p) + \varphi'(x, y, p, t),$$

$$\mathbf{V}_g = \overline{u}(y, p)\hat{i} + \mathbf{V}'_g(x, y, p, t), \qquad (20.2)$$

$$q_p = \overline{q}_p(y, p) + q'_p(x, y, p, t).$$

Substituting (20.2) into (20.1) and dropping terms that are quadratic in the perturbation variables gives

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) [\frac{1}{f_0}\nabla^2\varphi' + f_0\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial\varphi'}{\partial p}\right)] + v'_g\frac{\partial\overline{q}_p}{\partial y} = 0. \qquad (20.3)$$

Charney and Stern looked for modal solutions of the form

$$\varphi' = \Phi(y, p)e^{ik(x-ct)}, \tag{20.4}$$

where Φ is a complex function of y and p and c is a complex phase speed. Substitution of (20.4) into (20.3) gives

$$\frac{\partial^2 \Phi}{\partial y^2} + f_0^2 \frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial \Phi}{\partial p}\right) + \Phi\left(\frac{\partial \overline{q}_p/\partial y}{\overline{u} - c} - k^2\right) = 0. \tag{20.5}$$

As in section 18.2, we multiply by the complex conjugate of Φ and integrate over a domain that is infinite in y but bounded by rigid plates in p:

$$\int_{-\infty}^{\infty}\int_{p_1}^{p_0}\left[\Phi^*\frac{\partial^2 \Phi}{\partial y^2} + f_0^2\Phi^*\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial \Phi}{\partial p}\right) + |\Phi|^2\left(\frac{\partial \overline{q}_p}{\partial y}\frac{1}{\overline{u}-c} - k^2\right)\right]dp\,dy = 0, \tag{20.6}$$

where p_1 and p_0 are the pressures at the top and bottom boundary, respectively. We assume that as $y \to \pm\infty$, the geopotential perturbations or their meridional gradients vanish. Integrating (20.6) by parts and making use of this boundary condition gives

$$\int_{-\infty}^{\infty}\int_{p_1}^{p_0}\left\{\left|\frac{\partial \Phi}{\partial y}\right|^2 + \left(k^2 - \frac{\partial \overline{q}_p/\partial y}{\overline{u}-c}\right)|\Phi|^2 + \frac{f_0^2}{\mathcal{S}}\left|\frac{\partial \Phi}{\partial p}\right|^2\right\}dy\,dp$$

$$+ f_0^2\int_{-\infty}^{\infty}\left[\frac{\Phi^*}{\mathcal{S}}\frac{\partial \Phi}{\partial p}\right]\Big|_{p_1}^{p_0}dy = 0. \tag{20.7}$$

The last term in (20.7) involves geopotential perturbations at the two boundaries. For these, we use the Eady boundary condition given by (11.1). Linearizing this about the background state and using (20.4) gives

$$\frac{\partial \Phi}{\partial p} - \frac{\overline{\alpha}}{f_0\overline{\theta}(\overline{u}-c)}\frac{\partial \overline{\theta}}{\partial y}\Phi = 0 \text{ on } p = p_0, p_1. \tag{20.8}$$

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Substituting this into the last term of (20.7) gives

$$\int_{-\infty}^{\infty}\left[\int_{p_1}^{p_0}\left(\left|\frac{\partial\Phi}{\partial y}\right|^2+\left(k^2-\frac{\partial\overline{q}_p/\partial y}{\overline{u}-c}\right)|\Phi|^2+\frac{f_0^2}{\mathcal{S}}\left|\frac{\partial\Phi}{\partial p}\right|^2\right)dp\right.$$

$$\left.-\frac{\overline{\alpha}f_0\frac{\partial\overline{\theta}}{\partial y}}{\theta\mathcal{S}(\overline{u}-c)}|\Phi|^2\Big|_{p_1}^{p_0}\right]dy=0. \tag{20.9}$$

Since c is in general a complex number, the real and imaginary parts of (20.9) must

be satisfied independently. In particular, the imaginary part of (20.9) is

$$c_i\int_{-\infty}^{\infty}\left[\int_{p_1}^{p_0}\left(\frac{\partial\overline{q}_p/\partial y}{|\overline{u}-c|^2}|\Phi|^2\right)dp+\frac{\overline{\alpha}f_0\frac{\partial\overline{\theta}}{\partial y}}{\theta\mathcal{S}|\overline{u}-c|^2}|\Phi|^2\Big|_{p_1}^{p_0}\right]dy=0 \tag{20.10}$$

From this expression, it can be seen that for exponentially growing normal

modes $(c_i>0)$, one or more of the following must be true:

1. The meridional gradient of pseudo-potential vorticity, $\partial\overline{q}_p\partial y$, changes sign in

   the domain;

2. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, changes sign along one or both

   boundaries;

3. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, at the lower boundary (p_0) has

   the same sign as $\partial\overline{\theta}/\partial y$ at the upper boundary (p_1) and/or the opposite sign

   of the interior pseudo potential vorticity gradient, $\partial\overline{q}_p/\partial y$;

4. The meridional temperature gradient, $\partial\overline{\theta}/\partial y$, at the upper boundary (p_1) has

   the same sign as either or both the meridional temperature gradient at the

   lower boundary and the interior potential vorticity gradient.

In the Eady model, $\partial\overline{q}_p/\partial y=0$ but the temperature gradient has the same sign

at both boundaries, so the Charney-Stern necessary condition is satisfied. In the

Charney model, there is no temperature gradient at the upper boundary, but the temperature gradient at the lower boundary has the opposite sign as the interior pseudo-potential vorticity gradient, $\hat{\beta}$, so once again the necessary condition for instability is satisfied.

The Charney-Stern theorem may be interpreted as the requirement that at least two, counter-propagating (relative to the background flow) trains of Rossby waves must be supported by the fluid flow in order the normal mode instability to occur. One can also derive a Fjørtoft condition by taking the real part of (20.9); this shows, as in the Rayleigh instability problem, that the background flow must be configured so that the counter-propagating wave trains can become phase locked.

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