**12.864 Inference from Data and Models 7 April 2004**
**Problem Set No. 5 Due: 21 April 2004**

1. Consider the differential equation,

$$\frac{d^2\xi}{dt^2} + 2\xi(t) = q(t).$$

Discretize it in finite differences (your choice of how to do it) , and write it in the canonical statespace form for $\xi(n\Delta t), \Delta\ t = 0.2$. The discrete version of $q(t)$ is a simple zero-mean white-noise process of variance unity. An estimate of the intial conditions is $\xi\ (0) = 1,\ \xi\ (\Delta t) = 0$, where $\left\langle (\xi\ (0) - \xi\ (0))^2 \right\rangle =$ 1; $\left\langle (\xi\ (1) - \xi\ (\Delta t))^2 \right\rangle = 2$. There are no observations until $t = 20\Delta t$, when $y(20\Delta t) = \xi(20\Delta t) + n(20\Delta t) = 8 \pm 1$.

a. Using a Kalman filter code of your own devising, make a best-estimate of $\xi(n\Delta t), 0 \le n \le 22$ along with an uncertainty estimate of your result.

b. Using the RTS smoother make an estimate of $q(n\Delta t)$ in this same interval and of the actual initial conditions.

2. Using the method of Lagrange multipliers (adjoint method), re-solve problem 1b. Do the answers differ?

3. A state vector has two constant elements $\mathbf{x}(t) = [a, b]^T$. Measurements, $\mathbf{y}(t) = \mathbf{E}(t)\mathbf{x} + \mathbf{n}(t)$, are available, where $\mathbf{E}(t) = [1, t]$, $\langle \mathbf{n}(t) \rangle = 0, \left\langle \mathbf{n}(t)\mathbf{n}(t)^T \right\rangle = \mathbf{I}$. The measurements are, $\mathbf{y}(t)^T = [2.9624, 5.3273, 7.1746, 8.8133, 11.7258, 12.4117, 17.1832]$, $t = 1, 2, 3, ...$,and are obtained one at a time. Make a best estimate of $a, b$, and the corresponding uncertainty as a function of time as the data are received.

4. A model is described in its canonical form as,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1),\ \Delta t = 1,$$

where $\mathbf{A} = \left\{ \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right\}$, with initial condition $\mathbf{x}(0)^T = [1, 0.2]$, $\mathbf{P}(0) = \left\{ \begin{array}{cc} .01 & 0 \\ 0 & .01 \end{array} \right\}$ Measurements $\mathbf{y}(t) = \mathbf{E}\mathbf{x}(t) + \mathbf{n}(t)$ with the same statistics and values as in Problem 3. Here $\mathbf{E}(t) = \mathbf{E} = [1, 1]$. Find the best estimate of $\mathbf{x}(0)$ with its new uncertainty.