MIT 14.11

Mixed Strategy Matrix Form Games and Nash Equilibria

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1 Introduction

In the previous handout, we considered only pure strategies. However, in reality players are always able to randomly choose between their pure strategies, but our "available strategies" didn't allow for such mixing. More importantly, players may play his strategy against a player who is randomly drawn from a population consisting of heterogeneous strategies. How can we describe his payoffs in such a case when he is facing a "distribution" of strategies?

Consider the "matching pennies game." In this game each player plays one of two strategies, player 1 prefers if they play the same strategy, whereas player 2 prefers if they play opposite strategies. Notice that there are no pure Nash equilibrium.

$$\begin{array}{c c} & \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}$$

To address this concern, we will simply need to extend our definition of a strategy and a utility function, by assuming each player can choose a probability distribution over her strategies, and the player cares about her expected payoffs. The definition of Nash equilibrium also extends quite naturally, although the algorithm for finding a Nash equilibrium becomes a bit more complicated.

2 Mixed Strategy Games

<u>Definition 1.4</u>: A **mixed strategy extension** of a matrix form game $< \{S_i\}_{i=1,2}, \{U_i\}_{i=1,2} >$ is defined by the tuple $< \{\Delta S_i\}_{i=1,2}, \{\mathcal{U}_i\}_{i=1,2} >$ where:

- ΔS_i is the set of mixed strategies available to player i (probability distributions over the strategies S_i). Or,

  $$\Delta S_i = \{\alpha_i | \alpha_i : s_i \to [0,1] s.t. \sum_{s_i \in S_i} \alpha_i(s_i) = 1\}$$

  This is just a really technical way of saying, "the probability i places on each strategy." E.g.: for the matching pennies game, we can display the set of mixed strategies as

  $$\Delta S_i = \{\alpha_i = (h,t) | h + t = 1 \ \&\ h,t > 0\}$$

  One such strategy is $\alpha_i = (.5, .5)$, in which player i plays H half the time. By the way, sometimes people write $\alpha_i = H$. When they do this, they mean $\alpha_i = (1,0)$.

- $\mathcal{U}_i$ is the expected payoff of a mixed strategy $\alpha \in \Delta S$. It is the expected utility from this mixed strategy, which is just a weighted average of the payoffs:

  $$\mathcal{U}_i(\alpha) = \sum_{s \in S} U_i(s) \alpha_i(s)$$

  E.g.: if $\alpha_i = (.5, .5)$ for $i = 1, 2$, then $\mathcal{U}_1(\alpha) = \mathcal{U}_2(\alpha) = 0$.

Definition 1.5: A mixed strategy $\alpha = (\alpha_i, \alpha_{-i})$ is a **mixed strategy Nash equilibrium** if for each player i and every mixed strategy α_i of player i, the expected payoff to player i is greater than or equal to the expected payoff to player i of (α_i', α_{-i}). More concisely, α is a mixed strategy Nash equilibrium if $\forall i, \forall \alpha_i \in \Delta S_i$:

$$\mathcal{U}_i(\alpha_i, \alpha_{-i}) \geq \mathcal{U}_i(\alpha_i', \alpha_{-i}).$$

Next, we wish to find which mixed strategy players might choose as part of a Nash equilibrium. Start by recognizing the following. If i plays a mixing strategy in equilibrium, it must be the case that i is indifferent between the pure strategies she's mixing over. Otherwise, i would prefer to play the preferred pure strategy with probability 1. Moreover, if i is playing a mixed strategy as part of a Nash equilibrium, it must be the case that this mixed strategy is better than any pure strategies she's not mixing over. This implies that we can characterize the strategy i plays as part of a mixed Nash equilibrium by the pure strategies she mixes over.

Proposition 1.1: A mixed strategy α is a mixed Nash equilibrium if $\forall i \forall s_i$ such that $\alpha_i(s_i) > 0$, $\forall s_i'$

$$U_i(s_i, \alpha_{-i}) \geq U_i(s_i', \alpha_{-i})$$

The above proposition provides a simple algorithm for calculating mixed Nash equilibria. Which we will call the p-algorithm, which we will illustrate in an example.

$$\begin{array}{cc} & \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}$$

If player 1 is mixing between his two strategies, player two must assign probability p to H in such a way that player 1 gets the same expected utility for H and T

$$\mathcal{U}_1(H, p(H)) + (1-p)T) = \mathcal{U}_1(T, p(H) + (1-p)T)$$

so

$$pU_1(H,H) + (1-p)U_1(H,T) = pU_1(T,H) + (1-p)U_1(T,T)$$

plugging in from the payoff matrix, we get

$$p \cdot 1 + (1-p) \cdot (-1) = p(-1) + (1-p) \cdot 1$$

So $p = \frac{1}{2}$. The same can be done to find $q = \frac{1}{2}$, the probability that 1 plays H.

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