14.12 Game Theory – Final

12/14/2010

(FYI only; not proof read)

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have two hour and 50 minutes. Each question is 25 points. Good luck!

1. Consider a two player Bayesian game with the following payoff matrix

	R	S	P
R	$f(\theta_1), f(\theta_2)$	$f(\theta_1)+10, g(\theta_2)-10$	$f(\theta_1)-10, h(\theta_2)+10$
S	$g(\theta_1)-10, f(\theta_2)+10$	$g(\theta_1), g(\theta_2)$	$g(\theta_1)+10, h(\theta_2)-10$
P	$h(\theta_1)+10, f(\theta_2)-10$	$h(\theta_1)-10, g(\theta_2)+10$	$h(\theta_1), h(\theta_2)$

where $\theta_i \in \{0,1,2\}$ is privately known by player i and $f(0)=1$, $f(1)=f(2)=0$, $g(1)=1$, $g(0)=g(2)=0$, $h(2)=1$, and $h(0)=h(1)=0$. The functions f, g, and h are known and each pair (θ_1,θ_2) has probability 1/9.

(a) (5 points) Write this as a Bayesian game.

(b) (20 points) Find a Bayesian Nash equilibrium of this game. Verify that the strategy profile you identified is indeed a Bayesian Nash equilibrium.

2. There are two identical objects and three potential buyers, named 1, 2, and 3. Each buyer only needs one object and does not care which of the identical objects he gets. The value of the object for buyer i is v_i where (v_1,v_2,v_3) are independently and uniformly distributed on $[0,1]$. The objects are sold to two of the buyers through the following auction. Simultaneously, each buyer i submits a bid b_i, and the buyers who bid one of the two highest bids buy the object and pay their own bid. (The ties are broken by a coin toss.) That is, if $b_i > b_j$ for some j, i gets an object and pays b_i, obtaining the payoff of $v_i - b_i$; if $b_i < b_j$ for all j, the payoff of i is 0.

(a) (5 points) Write this as a Bayesian game.

(b) (20 points) Compute a symmetric Bayesian Nash equilibrium of this game in increasing differentiable strategies. (You will receive 15 points if you derive the correct equations without solving them.)

3. A state government wants to construct a new road. There are n construction firms. In order to decrease the cost of delay in completion of the road, the government wants to divide the road into $k < n$ segments and construct the segments simultaneously using different firms. The cost of delay for the public is $C_p = K/k$ for some constant $K > 0$. The cost of constructing a segment for firm i is c_i/k where $(c_1,\ldots,c_n)$ are independently and uniformly distributed on $[0,1]$, where c_i is privately known by firm i. The government hires the firms through the following procurement auction.

$k+1$**st-price Procurement Auction** Simultaneously, each firm i submits a bid b_i and each of the firms with the **lowest** k bids wins one of the segments. Each winning firm is paid the lowest $k+1$st bid as the price for the construction of the segment. The ties are broken by a coin toss.

The payoff of a winning firm is the price paid minus its cost of constructing a segment, and the payoff of a losing firm is 0. For example, if $k = 2$ and the bids are $(0.1, 0.2, 0.3, 0.4)$, then firms 1 and 2 win and each is paid 0.3, resulting in payoff vector $(0.3 - c_1/2, 0.3 - c_2/2, 0, 0)$.

(a) (10 points) For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

(b) (10 points) Assume that each winning firm is to pay $\beta \in (0, 1)$ share of the price to the local mafia. (In the above example it pays 0.3β to the mafia and keep $0.3(1 - \beta)$ for itself.) For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

(c) (5 points) Assuming that the government minimizes the sum of C_P and the total price it pays for the construction, find the condition for the optimal k for the government in parts (a) and (c). Show that the optimal k in (c) is weakly lower than the optimal k in (a). Briefly interpret the result. [Hint: the expected value of the $k + 1$st lowest cost is $(k + 1) / (n + 1)$.]

4. **Stage Game:** Alice and Bob simultaneously choose contributions $a \in [0, 1]$ and $b \in [0, 1]$, respectively, and get payoffs $u_A = 2b - a$ and $u_B = 2a - b$, respectively.

(a) (5 points) Find the set of rationalizable strategies in the Stage Game above.

(b) (10 points) Consider the infinitely repeated game with the Stage Game above and with discount factor $\delta \in (0, 1)$. For each δ, find the maximum (a^*, b^*) such that there exists a subgame-perfect equilibrium of the repeated game in which Alice and Bob contribute a^* and b^*, respectively, on the path of equilibrium.

(c) (10 points) In part (b), now assume that at the beginning of each period t one of the players (Alice at periods $t = 0, 2, 4, \ldots$ and Bob at periods $t = 1, 3, 5, \ldots$) offers a stream of contributions $\vec{a} = (a_t, a_{t+1}, \ldots)$ and $\vec{b} = (b_t, b_{t+1}, \ldots)$ for Alice and Bob, respectively, and the other player accepts or rejects. If the offer is accepted then the game ends leading the automatic contributions $\vec{a} = (a_t, a_{t+1}, \ldots)$ and $\vec{b} = (b_t, b_{t+1}, \ldots)$ from period t on. If the offer is rejected, they play the Stage Game and proceed to the next period. Find (a_A, b_A), (a_B, b_B), and $\left(\hat{a}, \hat{b}\right)$ such that the following is a subgame-perfect equilibrium:

s^* : When it is Alice's turn, Alice offers $(a_A, a_A, \ldots)$ and $(b_A, b_A, \ldots)$ and Bob accepts an offer $(\vec{a}, \vec{b})$ if and only if $(1 - \delta)\left[2a_t - b_t + \delta\left(2a_{t+1} - b_{t+1}\right) + \cdots\right] \geq 2a_A - b_A$. When it is Bob's turn, Bob offers $(a_B, a_B, \ldots)$ and $(b_B, b_B, \ldots)$ and Alice accepts an offer $(\vec{a}, \vec{b})$ if and only if $(1 - \delta)\left[2b_t - a_t + \delta\left(2b_{t+1} - a_{t+1}\right) + \cdots\right] \geq 2b_A - a_A$. If there is no agreement, in the stage game they play $\left(\hat{a}, \hat{b}\right)$.

Verify that s^* is a subgame perfect equilibrium for the values that you found. (If you find it easier, you can consider only the constant streams of contributions $\vec{a} = (a, a, \ldots)$ and $\vec{b} = (b, b, \ldots)$.)

14.12 Economic Applications of Game Theory
Fall 2012