Chapter 10

Application: Negotiation

Negotiation is an essential aspect of social and economic interaction. The states negotiate their borders with their neighbors; the legislators negotiate the laws that they make; defendants negotiate a settlement with the prosecutors or the plaintiffs in the courts; workers negotiate their salaries with their employers; the families negotiate their spending and maintenance of the household with each other, and even some students try to negotiate their grades with their professor. Despite its central importance, negotiations were presumed to be outside of the purview of economic analysis until the emergence of game theory. Today there are many game theoretical models of bargaining. These notes apply backward induction to three important bargaining games. The first one considers congressional bargaining. It abstracts away from the back-room deals that lead to the proposed bills and focus on the way legislators vote between various alternatives. The second model considers pretrial negotiation in law. The third one is a general model of bargaining that can be applied to many different settings in economics.

10.1 Congressional Bargaining—Voting with a Binary Agenda

In the US Congress, when a new bill introduced, there are often other alternative proposals, such as amendments, amendments to amendments, substitute bills, amendments to substitute bills, etc. There are rules of the Congress that determine the order in which these proposals, or "alternatives", are voted against each other, eventually leading to a

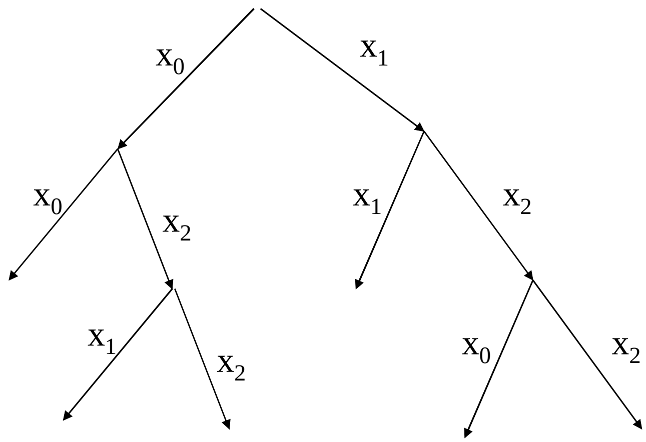

Figure 10.1: A binary agenda

final bill. In the final vote, the final bill, which may not be the original one, passes, or fails, in which case the status quo prevails. For example, if there is a bill, an amendment, and the status quo, first they vote between the bill and the amendment, then they vote between the winner of the previous vote and the status quo. These rules and the available proposals lead to a "binary" agenda; it is *binary* because in any session *two* alternatives are voted against each other.

Let $\{1, \ldots, 2n+1\}$ be the set of players and $\{x_0, \ldots, x_m\}$ be the set of alternatives. Each player has a strict preference ordering for the set of alternatives. There is a fixed binary agenda, and assume that all of these are commonly known.

To solve this game, we start from a last vote (a vote after which there is no further voting). We assume that each player votes according to his preference. The alternative that gets $n+1$ or more votes wins. We then truncate the game by replacing the vote with the winning alternative. We proceed in this way until there is only one alternative.

For example, consider three players, namely 1, 2, and 3, and three alternatives, namely x_0, x_1, and x_2. The agenda is as in Figure 10.1. According to the agenda, x_0 and x_1 are voted against each other first; the winner is voted against x_2 next. If the winner defeats x_2 as well, then it is implemented; otherwise x_2 (the winner of the second vote) is voted against the loser of the first vote and the winner of this vote is implemented.

The preference ordering of the three players is as follows:

1	2	3
x_0	x_2	x_1
x_1	x_0	x_2
x_2	x_1	x_0

where the higher-ranked alternatives are placed in the higher rows.

Consider the branch on the left first. In the last vote, which is between x_1 and x_2, every player vote his better alternative according to the table. Players 1 and 3 vote for x_1, and Player 2 votes for x_2. In this vote x_1 beats x_2. Now consider the preceding vote, between x_0 and x_2. Now, everyone knows that if x_2 wins, in the next round x_1 will be implemented. Hence, a vote for x_2 is simply a vote for x_1. Hence, in the backward induction, the final vote is replaced with its winner, namely x_1. Those who prefer x_0 to x_1, who are players 1 and 2, vote for x_0, and the other player, who prefers x_1 to x_0, votes for x_2. In this vote, x_0 wins.

Now consider the right branch. In the last round, between x_0 and x_2, Player 1 votes for x_0, and 2 and 3 vote for x_2, resulting in the winning of x_2. Hence, in the backward induction, the last round is replaced by x_2. In the previous vote between x_1 and x_2, if x_1 wins it is implemented, and if x_2 wins it will be implemented (after defeating x_0, which will happen). Then, each player votes according to his true preference: players 1 and 3 for x_1, and Player 2 for x_2. Alternative x_1 wins. Therefore, on the right branch, x_1 wins.

Finally, at the very first vote, between x_0 and x_1, the players know that the winning alternative will be implemented in the future. Hence, everybody votes according to his original preferences and x_0 wins.

An interesting phenomenon is called a *killer amendment* or a *poison pill*. Suppose that we have a bill x_1 that is preferred by a majority of the legislators to the status quo, x_0. If the bill is voted against the status quo, it will pass. A poison pill or a killer amendment is an amendment x_2 that is worse than the status quo, x_0, according to a majority. Recall that the amendment x_2 is first voted against the bill x_1 and the winner is finally voted against the status quo x_0. If the amendment passes, then it will fail in the last round, and the status quo will be kept. Hence the term *killer amendment*.

Note that according to backward induction, a killer amendment is defeated in the first round (assuming that a majority prefers x_1 to x_0). This is because if x_2 defeats x_1, in the next round x_0 will be implemented. Hence, in the first round a vote for x_2 is a vote for the status quo, x_0. Then, the players who prefer the status quo, x_0, to the bill will vote for the amendment, x_2, and the players who prefer the bill, x_1, to the status quo will vote for the bill. Since the latter group is a majority, x_1 defeats the amendment in the first round.

But poison pills and killer amendments are frequently introduced and sometimes they defeat the original bill (and eventually are defeated by the status quo). A famous example to this is DePew amendment to the "17th amendment to the constitution" in 1912. Here, the 17th amendment, x_1, requires the senators to be elected by the statewide popular vote. This bill was supported by the (Southern) Democrats and half of the Republicans, making up the two thirds of the congress. The DePew amendment, x_2, required that these elections be monitored by the federal government. Each Republican slightly prefers x_2 to x_1, so the proponent Republicans' ordering is $x_2 \succ x_1 \succ x_0$ and the opposing Republicans' ordering is $x_0 \succ x_2 \succ x_1$, where x_0 is the status quo. But the federal oversight of the state elections is unacceptable to the southern Democrats for obvious reasons: $x_1 \succ x_0 \succ x_2$. Notice that "opposing Republicans" and Democrats, which is about the two thirds of the legislators, prefer the status quo to the DePew amendment. Hence, the DePew amendment is a killer amendment. According to our analysis it should be defeated in the first round, and the original bill, the 17th amendment, should eventually pass. But this did *not* happen. The DePew amendment killed the bill.

Why does this happen? It would be too naive to think that a legislator is so myopic that he cannot see one step ahead and fails to recognize a killer amendment. Sometimes, legislators might not know the preferences of the other legislators. After all, these preferences are elicited in these elections. In that case, the backward induction analysis above is not valid and needs to be modified. Of course, in that case, an amendment may defeat the bill (because of the proponents who think that it has enough support for an eventual passage) but later be defeated in the final vote because of the lack of sufficient support (which was not known in the first vote). But mostly, the killer amendments are introduced intentionally, and the legislators have a clear idea about the preferences. Even in that case, a killer amendment can pass, not because of the stupidity of the proponents of the original bill, but because their votes against the amendment can be exploited by their opponents in the upcoming elections when the voters are not informed about the details of these bills.

The moral of the story is that it is not enough that your analysis is correct. You must also be analyzing the correct game. You will learn the first task in the Game Theory class; for the second, and more important, task of considering the correct game, you need to look at the underlying facts of the situation.

10.2 Pre-trial Negotiations

Consider two players: a Plaintiff and a Defendant. The Plaintiff suffers a loss due to the negligence of the Defendant, and he is suing her now. The court date is set at date $2n+1$. It is known that if they go to court, the Judge will order the Defendant to pay $J>0$ to the Plaintiff. But the litigation is very costly. For example, in the US, 95% of cases are settled without going to court. In order to avoid the legal costs, the Plaintiff and the Defendant are also negotiating an out of court settlement. The negotiation follows the following protocol.

- At each date $t \in \{1, 3, \ldots, 2n-1\}$, if they have not yet settled, the Plaintiff offers a settlement s_t,
- and the Defendant decides whether to accept or reject it. If she accepts, the game ends with the Defendant paying s_t to the Plaintiff; the game continues otherwise.
- At dates $t \in \{2, 4, \ldots, 2n\}$, the Defendant offers a settlement s_t,
- and the Plaintiff decides whether to accept the offer, ending the game with the Defendant paying s_t to the Plaintiff, or to reject it and continue.
- If they do not reach an agreement at the end of period $t = 2n$, they go to court, and the Judge orders the Defendant to pay $J > 0$ to the Plaintiff.

The Plaintiff pays his lawyer c_P for each day they negotiate and an extra C_P if they go to court. Similarly, the Defendant pays her lawyer c_D for each day they negotiate and an extra C_D if they go to court. Each party tries to maximize the expected amount of money he or she has at the end of the game.

The backward induction analysis of the game as follows. The payoff from going to court for the Plaintiff is

$$J - C_P - 2nc_P.$$

If he accepts the settlement offer s_{2n} of the Defendant at date $2n$, his payoff will be

$$s_{2n} - 2nc_P.$$

Hence, if $s_{2n} > J - C_P$, he must accept the offer, and if $s_{2n} < J - C_P$, he must reject the offer. If $s_{2n} = J - C_P$, he is indifferent between accepting and rejecting the offer. Assume that he accepts that offer, too.[1] To sum up, he accepts an offer s_{2n} if and only if $s_{2n} \geq J - C_P$.

What should the Defendant offer at date $2n$? Given the behavior of Plaintiff, her payoff from s_{2n} is

$$\begin{aligned} -s_{2n} - 2nc_D \text{ if } s_{2n} &\geq J - C_P \\ -J - C_D - 2nc_D \text{ if } s_{2n} &< J - C_P. \end{aligned}$$

This is because, if the offer is rejected, they will go to court. Notice that when $s_{2n} = J - C_P$, her payoff is $-J + C_P - 2nc_D$, and offering anything less would cause her to lose $C_D + C_P$. Her payoff is plotted in Figure 10.2. Therefore, the Defendant offers

$$s_{2n} = J - C_P$$

at date $2n$.

Now at date $2n-1$, the Plaintiff offers a settlement s_{2n-1} and the Defendant accepts or rejects the offer. If she rejects the offer, she will get the payoff from settling for $s_{2n} = J - C_P$ at date $2n$, which is

$$-J + C_P - 2nc_D.$$

If she accepts the offer, she will get

$$-s_{2n-1} - (2n-1)\, c_D.$$

[1] In fact, he must accept $s_{2n} = J - C_P$ in equilibrium. For, if he doesn't accept it, the best response of the Defendant will be empty, inconsistent with an equilibrium. (Any offer $s_{2n} = J - C_P + \epsilon$ with $\epsilon > 0$ will be accepted. But for any offer $s_{2n} = J - C_P + \epsilon$, there is a better offer $s_{2n} = J - C_P + \epsilon/2$, which will also be accepted.)

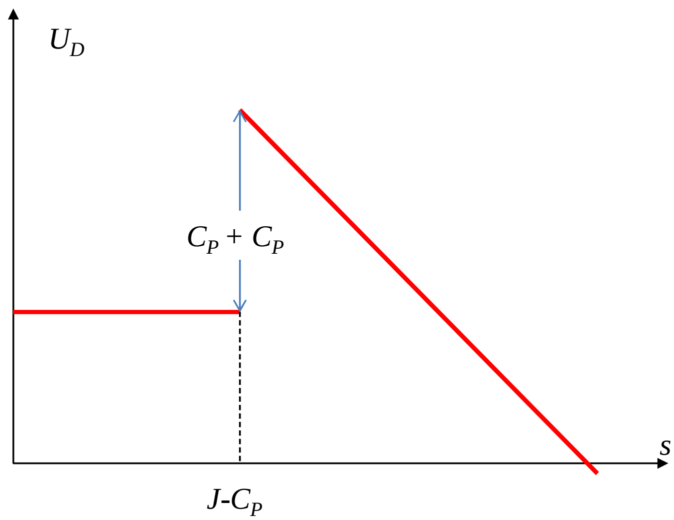

Figure 10.2: Payoff of Defendant from her offer at the last period

Hence, she will accept the offer if and only if the last expression is greater than or equal to the previous one, i.e.,

$$s_{2n-1} \leq J - C_P + c_D.$$

Then, the Plaintiff will offer the highest acceptable settlement (to the Defendant):

$$s_{2n-1} = J - C_P + c_D.$$

In summary, since the Plaintiff is making an offer, he offers the settlement amount of next date plus the cost of negotiating one more day for the Defendant.

Let us apply the backward induction one more step. At date $2n-2$, the Defendant offers a settlement s_{2n-2} and the Plaintiff accepts or rejects the offer. If he rejects the offer, he will get the payoff from settling for $s_{2n-1} = J - C_P + c_D$ at date $2n-1$, which is

$$s_{2n-1} - (2n-1)\, c_P = J - C_P + c_D - (2n-1)\, c_P.$$

If he accepts the offer, he will get

$$s_{2n-2} - (2n-2)\, c_P.$$

Hence, he will accept the offer if and only if the last expression is greater than or equal to the previous one, i.e.,

$$s_{2n-2} \geq s_{2n-1} - c_P = J - C_P + c_D - c_P.$$

Then, the Defendant offers the highest acceptable settlement (to the Plaintiff):

$$s_{2n-1} = s_{2n-1} - c_P = J - C_P + c_D - c_P.$$

In summary, since the Defendant is making an offer, she offers the settlement amount of next date *minus* the cost of negotiating one more day for the Plaintiff.

Now the pattern is clear. At any odd date t, the Defendant accepts an offer s_t if and only if $s_t \leq s_{t+1} + c_D$, and the Plaintiff offers

$$s_t = s_{t+1} + c_D. \qquad (t \text{ is odd})$$

At any even date t, the Plaintiff accepts an offer s_t if and only if $s_t \geq s_{t+1} - c_P$, and the Defendant offers

$$s_t = s_{t+1} - c_P \qquad (t \text{ is even}).$$

The solution to the above difference equation is

$$s_t = \begin{cases} J - C_P + (n - t/2)(c_D - c_P) & \text{if } t \text{ is even} \\ J - C_P + (n - (t+1)/2)(c_D - c_P) + c_D & \text{if } t \text{ is odd.} \end{cases}$$

Recall from the lecture that the solution is substantially different if the order of the proposers is changed (see the slides). This is because at the last day, the cost of delaying the agreement is quite high (the cost of going to court), and the party who accepts or rejects the offer is willing to accepts a wide range of offers. Hence, the last proposer has a great advantage.

10.3 Sequential Bargaining

Imagine that two players own a dollar, which they can use only after they decide how to divide it. Each player is risk-neutral and discounts the future exponentially. That is, if a player gets x dollar at day t, his payoff is $\delta^t x$ for some $\delta \in (0,1)$. The set of all feasible divisions is $D = \left\{(x,y) \in [0,1]^2 \,|x + y \leq 1\right\}$. The players are bargaining over the division of the dollar by making offers and counteroffers, as it will be clear momentarily. We want to apply backward induction to this game in order to understand when the parties will reach an agreement and what the terms of the agreement will be.

First consider the following simplified model with only two rounds of negotiations. In the first day, Player 1 makes an offer $(x_1, y_1) \in D$. Then, knowing what has been offered, Player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs (x_1, y_1). If he rejects the offer, then they wait until the next day, when Player 2 makes an offer $(x_2, y_2) \in D$. Now, knowing what Player 2 has offered, Player 1 accepts or rejects the offer. If Player 1 accepts the offer, the offer is implemented, yielding payoffs $(\delta x_2, \delta y_2)$. If Player 2 rejects the offer, then the game ends, when they lose the dollar and get payoffs (0,0).

The backward induction analysis of this simplified model is as follows. On the second day, if Player 1 rejects the offer, he gets 0. Hence, he accepts any offer that gives him more than 0, and he is indifferent between accepting and rejecting any offer that gives him 0. As we have seen in the previous section, he accepts the offer (0,1) in equilibrium. Then, on the second day, Player 2 would offer (0,1), which is the best Player 2 can get. Therefore, if they do not agree on the first day, then Player 2 takes the entire dollar on the second day, leaving Player 1 nothing. The value of taking the dollar on the next day for Player 2 is δ. Hence, on the first day, Player 2 accepts any offer that gives him more than δ, rejects any offer that gives him less than δ, and he is indifferent between accepting and rejecting any offer that gives him δ. As above, assume that Player 2 accepts the offer $(1-\delta, \delta)$. In that case, Player 1 offers $(1-\delta, \delta)$, which is accepted. Could Player 1 receive more than $1-\delta$? If he offered anything that is better than $1-\delta$ for himself, his offer would necessarily give less than δ to Player 2, and Player 2 would reject the offer. In that case, the negotiations would continue to the next day and he would receive 0, which is clearly worse than $1-\delta$.

Now, consider the game in which the game above is repeated n times. That is, if they have not yet reached an agreement by the end of the second day, on the third day, Player 1 makes an offer $(x_3, y_3) \in D$. Then, knowing what has been offered, Player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^2 x_3, \delta^2 y_3\right)$. If he rejects the offer, then they wait until the next day, when Player 2 makes an offer $(x_4, y_4) \in D$. Now, knowing what Player 2 has offered, Player 1 accepts or rejects the offer. If Player 1 accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^3 x_4, \delta^3 y_4\right)$. If Player 1 rejects the offer, then they go to the 5th day... And this goes on like this until the end of day $2n$. If they have not yet agreed at the end of that day, the game ends, they lose the dollar and get payoffs (0,0).

Application of backward induction to this game results in the following strategy profile. At any day $t = 2n - 2k$ (k is a non-negative integer), Player 1 accepts any offer (x, y) with

$$x \geq \frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta}$$

and rejects any offer (x, y) with

$$x < \frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta}$$

Player 2 offers

$$(x_t, y_t) = \left(\frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta}, 1 - \frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta}\right) \equiv \left(\frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta}, \frac{1 + \delta^{2k+1}}{1 + \delta}\right).$$

And at any day $t - 1 = 2n - 2k - 1$, Player 2 accepts an offer (x, y) iff

$$y \geq \frac{\delta\left(1 + \delta^{2k+1}\right)}{1 + \delta}$$

and Player 1 offers

$$(x_{t-1}, y_{t-1}) = \left(1 - \frac{\delta\left(1 + \delta^{2k+1}\right)}{1 + \delta}, \frac{\delta\left(1 + \delta^{2k+1}\right)}{1 + \delta}\right) \equiv \left(\frac{1 - \delta^{2k+2}}{1 + \delta}, \frac{\delta\left(1 + \delta^{2k+1}\right)}{1 + \delta}\right).$$

We can prove this is indeed the equilibrium given by backward induction using mathematical induction on k. (That is, we first prove that it is true for $k = 0$; then assuming that it is true for some $k - 1$, we prove that it is true for k.)

Proof. Note that for $k = 0$, we have the last two periods, identical to the 2-period example we analyzed above. Letting $k = 0$, we can easily check that the behavior described here is the same as the equilibrium behavior in the 2-period game. Now, assume that, for some $k - 1$ the equilibrium is as described above. That is, at the beginning of date $t + 1 := 2n - 2(k - 1) - 1 = 2n - 2k + 1$, Player 1 offers

$$(x_{t+1}, y_{t+1}) = \left(\frac{1 - \delta^{2(k-1)+2}}{1 + \delta}, \frac{\delta\left(1 + \delta^{2(k-1)+1}\right)}{1 + \delta}\right) = \left(\frac{1 - \delta^{2k}}{1 + \delta}, \frac{\delta\left(1 + \delta^{2k-1}\right)}{1 + \delta}\right);$$

and his offer is accepted. At date $t = 2n - 2k$, Player 1 accepts an offer iff the offer is at least as good as having $\frac{1-\delta^{2k}}{1+\delta}$ the next day, which is worth $\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}$. Therefore, he accepts an offer (x, y) iff

$$x \geq \frac{\delta\left(1 - \delta^{2k}\right)}{1 + \delta};$$

as in the strategy profile above. In that case, the best Player 2 can do is to offer

$$(x_t, y_t) = \left(\frac{\delta \left(1-\delta^{2k}\right)}{1+\delta}, 1 - \frac{\delta \left(1-\delta^{2k}\right)}{1+\delta} \right) = \left(\frac{\delta \left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right).$$

This is because any offer that gives Player 2 more than y_t will be rejected in which case Player 2 will get

$$\delta y_{t+1} = \frac{\delta^2 \left(1+\delta^{2k-1}\right)}{1+\delta} < y_t.$$

In summary, at t, Player 2 offers (x_t, y_t); and it is accepted. Consequently, at $t-1$, Player 2 accepts an offer (x, y) iff

$$y \geq \delta y_t = \frac{\delta \left(1+\delta^{2k+1}\right)}{1+\delta}.$$

In that case, at $t-1$, Player 1 offers

$$(x_{t-1}, y_{t-1}) \equiv (1-\delta y_t, \delta y_t) = \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta \left(1+\delta^{2k+1}\right)}{1+\delta} \right),$$

completing the proof. ■

Now, let $n \to \infty$. At any odd date t, Player 1 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta \left(1+\delta^{2k+1}\right)}{1+\delta} \right) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right);$$

and any even date t, Player 2 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left(\frac{\delta \left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right) = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta} \right);$$

The offers are barely accepted.

10.4 Exercises with Solutions

1. Consider two agents $\{1, 2\}$ owning one dollar which they can use only after they divide it. Each player's utility of getting x dollar at t is $\delta^t x$ for $\delta \in (0, 1)$. Given any $n > 0$, consider the following n-period symmetric, random bargaining model. Given any date $t \in \{0, 1, \ldots, n-1\}$, we toss a fair coin; if it comes Head (which comes with probability 1/2), we select player 1; if it comes Tail, we select player

2. The selected player makes an offer $(x, y) \in [0, 1]^2$ such that $x + y \leq 1$. Knowing what has been offered, the other player accepts or rejects the offer. If the offer (x, y) is accepted, the game ends, yielding payoff vector $(\delta^t x, \delta^t y)$. If the offer is rejected, we proceed to the next date, when the same procedure is repeated, except for $t = n - 1$, after which the game ends, yielding (0,0). The coin tosses at different dates are stochastically independent. And everything described up to here is common knowledge.

(a) Compute the subgame perfect equilibrium for $n = 1$. What is the value of playing this game for a player? (That is, compute the expected utility of each player before the coin-toss, given that they will play the subgame-perfect equilibrium.)

Solution: If a player rejects an offer, he will get 0, hence he will accept any offer that gives him at least 0. (He is indifferent between accepting and rejecting an offer that gives him exactly 0; but rejecting such an offer is inconsistent with an equilibrium.) Hence, the selected player offers 0 to his opponent, taking entire dollar for himself; and his offer will be accepted. Therefore, in any subgame perfect equilibrium, the outcome is (1,0) if it comes Head, and (0, 1) if it comes Tail. The expected payoffs are

$$V = \frac{1}{2}(1,0) + \frac{1}{2}(0,1) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

(b) Compute the subgame perfect equilibrium for $n = 2$. Compute the expected utility of each player before the <u>first</u> coin-toss, given that they will play the subgame-perfect equilibrium.

Solution: In equilibrium, on the last day, they will act as in part (a). Hence, on the first day, if a player rejects the offer, the expected payoff of each player will be $\delta \cdot 1/2 = \delta/2$. Thus, he will accept an offer if an only if it gives him at least $\delta/2$. Therefore, the selected player offers $\delta/2$ to his opponent, keeping $1 - \delta/2$ for himself, which is more than $\delta/2$, his expected payoff if his offer is rejected. Therefore, in any subgame perfect equilibrium, the outcome is $(1 - \delta/2, \delta/2)$ if it comes Head, and $(\delta/2, 1 - \delta/2)$ if it comes Tail. The expected payoff of each player before the first coin toss is

$$\frac{1}{2}(1-\delta/2)+\frac{1}{2}(\delta/2)=\frac{1}{2}.$$

(c) What is the subgame perfect equilibrium for $n \geq 3$.

Solution: Part (b) suggests that, if expected payoff of each player at the beginning of date $t+1$ is $\delta^{t+1}/2$, the expected payoff of each player at the beginning of t will be $\delta^{t}/2$. [Note that in terms of dollars these numbers correspond to $\delta/2$ and $1/2$, respectively.] Therefore, the equilibrium is follows: At any date $t < n-1$, the selected player offers $\delta/2$ to his opponent, keeping $1-\delta/2$ for himself; and his opponent accepts an offer iff he gets at least $\delta/2$; and at date $n-1$, a player accepts any offer, hence the selected player offers 0 to his opponent, keeping 1 for himself. [You should be able to prove this using mathematical induction and the argument in part (b).]

2. [Midterm 1, 2002] Consider two players A and B, who own a firm and want to dissolve their partnership. Each owns half of the firm. The value of the firm for players A and B are v_A and v_B, respectively, where $v_A > v_B > 0$. Player A sets a price p for half of the firm. Player B then decides whether to sell his share or to buy A's share at this price, p. If B decides to sell his share, then A owns the firm and pays p to B, yielding playoffs $v_A - p$ and p for players A and B, respectively. If B decides to buy, then B owns the firm and pays p to A, yielding playoffs p and $v_B - p$ for players A and B, respectively. All these are common knowledge. Applying backward induction, find a Nash equilibrium of this game.

Solution: Given any price p, the best response of B is

$$\begin{cases} \text{buy} & \text{if } v_B - p > p, \text{ i.e., if } p < v_B/2; \\ \text{sell} & \text{if } p > v_B/2; \\ \{\text{buy, sell}\} & \text{if } p = v_B/2. \end{cases}$$

In equilibrium, B must be selling at price $p = v_B/2$. This is because, if he were buying, then the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p \leq v_B/2; \\ v_A - p & \text{if } p > v_B/2, \end{cases}$$

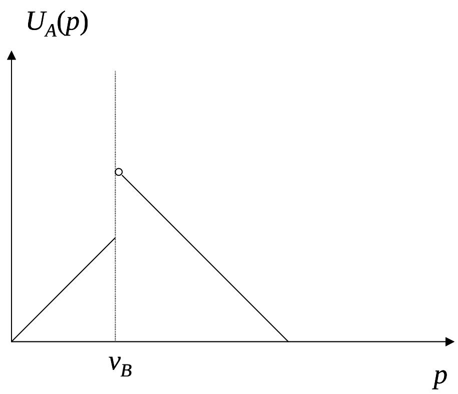

Figure 10.3:

which can be depicted as in Figure 10.3. Then, no price could maximize the payoff of A, inconsistent with equilibrium (where A maximizes his payoff given what he anticipates). Hence, the equilibrium strategy of B must be

$$\begin{cases} \text{buy} & \text{if } p < v_B/2; \\ \text{sell} & \text{if } p \geq v_B/2. \end{cases}$$

In that case, the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p < v_B/2; \\ v_A - p & \text{if } p \geq v_B/2, \end{cases}$$

which can be depicted as in Figure 10.4.This function is maximized at $p = v_B/2$. Player A sets the price as $p = v_B/2$.

3. [Midterm 1, 2006] Paul has lost his left arm due to complications in a surgery. He is suing the Doctor.

   - The court date is set at date $2n + 1$. It is known that if they go to court, the judge will order the Doctor to pay $J > 0$ to Paul.
   - They negotiate for a settlement before the court. At each date $t \in \{1, 3, \ldots, 2n - 1\}$, if they have not yet settled, Paul offers a settlement s_t, and the Doctor decides whether to accept or reject it. If she accepts, the game ends with the Doctor paying s_t to Paul; game continues otherwise. At dates $t \in \{2, 4, \ldots, 2n\}$, the

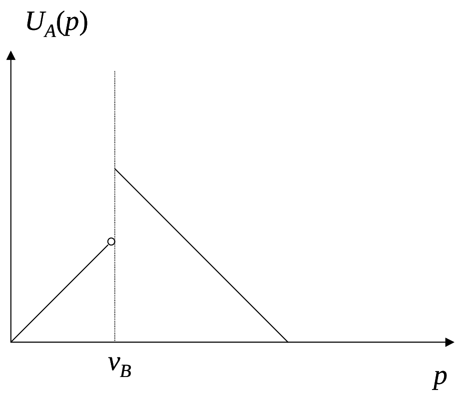

Figure 10.4:

Doctor offers a settlement s_t, and Paul decides whether to accept the offer, ending the game with Doctor paying s_t to Paul, or to reject it and continue.

- Paul pays his lawyer only a share of the money he gets from the Doctor. He pays $(1-\alpha)\, s_t$ if they settle at date t; $(1-\beta)\, J$ if they go to court, where $0 < \beta < \alpha < 1$. The Doctor pays her lawyer c for each day they negotiate and an extra C if they go to court.
- Each party tries to maximize the expected amount of money he or she has at the end of the game.

(a) (10 pts) For $n = 2$, apply backward induction to find an equilibrium of this game. (If you answer part (b) correctly, you don't need to answer this part.)

(b) (15 pts) For any n, apply backward induction to find an equilibrium of this game.

Answer: At date $2n+1$, Paul gets J from the doctor and pays $(1-\beta)\, J$ to his lawyer, netting βJ. Now at date $2n$, if he accepts s_{2n}, he will pay $(1-\alpha)\, s_{2n}$ to his lawyer, receiving αs_{2n}. Hence, he will accept s_{2n} iff $s_{2n} \geq (\beta/\alpha)\, J$. The doctor will offer

$$s_{2n} = (\beta/\alpha)\, J,$$

instead of going to court and paying $J > (\beta/\alpha)\, J$ to Paul and an extra C to her lawyer. Now, at $2n-1$, the Doctor will accept s_{2n-1} iff $s_{2n-1} \leq (\beta/\alpha)\, J + c$, as she would pay $(\beta/\alpha)\,J$ to Paul next day and an extra c to her lawyer. Paul will then offer

$$s_{2n-1} = (\beta/\alpha)\,J + c,$$

as the settlement will be only $(\beta/\alpha)\,J$ next day. He nets $\alpha s_{2n-1} = \beta J + \alpha c$ for himself. Now, at $2n-2$, Paul will accept an offer s_{2n-2} iff $s_{2n-2} \geq s_{2n-1} = (\beta/\alpha)\,J + c$, for he could settle for s_{2n-1} next day. (Note that offer gives him αs_{2n-2} and rejection gives him s_{2n-1}.) Therefore, the Doctor would offer him

$$s_{2n-2} = s_{2n-1}.$$

The pattern is now clear. At any odd date t, the Doctor accepts an offer iff $s_t \leq s_{t+1} + c$, and Paul offers

$$s_t = s_{t+1} + c \qquad (t \text{ is odd}).$$

At any even date t, Paul accepts an offer iff $s_t \geq s_{t+1}$, and the Doctor offers

$$s_t = s_{t+1} \qquad (t \text{ is even}).$$

This much is more or less enough for an answer. To be complete, note that the solution to the above equations is

$$s_t = \begin{cases} \frac{\beta}{\alpha}J + \frac{2n+1-t}{2}c & \text{if } t \text{ is odd} \\ \frac{\beta}{\alpha}J + \frac{2n-t}{2}c & \text{if } t \text{ is even} \end{cases}.$$

At the beginning, Paul offers $s_1 = (\beta/\alpha)\,J + nc$, which is barely accepted by the Doctor.

(c) (10 pts) Suppose now that with probability 1/2 the Judge may become sick on the court date and a Substitute Judge decide the case in the court. The Substitute Judge is sympathetic to doctors and will dismiss the case. In that case, the Doctor does not pay anything to Paul. (With probability 1/2, the Judge will order the Doctor to pay J to Paul.) How would your answer to part (b) change?

Answer: The expected payment in the court is now

$$J' = \frac{1}{2} \cdot J + \frac{1}{2} \cdot 0 = J/2.$$

Hence, we simply replace J with $J/2$. That is,

$$s_t = \begin{cases} \frac{\beta}{2\alpha}J + \frac{2n+1-t}{2}c & \text{if } t \text{ is odd} \\ \frac{\beta}{2\alpha}J + \frac{2n-t}{2}c & \text{if } t \text{ is even} \end{cases}.$$

10.5 Exercises

1. [Final, 2000] Consider a legal case where a plaintiff files a suit against a defendant. It is common knowledge that, when they go to court, the defendant will have to pay \$1000,000 to the plaintiff, and \$100,000 to the court. The court date is set 10 days from now. Before the court date plaintiff and the defendant can settle privately, in which case they do not have the court. Until the case is settled (whether in the court or privately) for each day, the plaintiff and the defendant pay \$2000 and \$1000, respectively, to their legal team. To avoid all these costs plaintiff and the defendant are negotiating in the following way. In the first day demands an amount of money for the settlement. If the defendant accepts, then he pays the amount and they settle. If he rejects, then he offers a new amount. If the plaintiff accepts the offer, they settle for that amount; otherwise the next day the plaintiff demands a new amount; and they make offers alternatively in this fashion until the court day. Players are risk neutral and <u>do not</u> discount the future. Applying backward induction, find a Nash equilibrium.

2. We have a Plaintiff and a Defendant, who is liable for a damage to the Plaintiff. If they go to court, then with probability 0.1 the Plaintiff will win and get a compensation of amount \$100,000 from the Defendant; if he does not win, there will be no compensation. Going to court is costly: if they go to court, each of the Plaintiff and Defendant will pay \$20,000 for the legal costs, independent of the outcome in the court. Both the Plaintiff and the Defendant are risk-neutral, i.e., each maximizes the expected value of his wealth.

   (a) Consider the following scenario: The Plaintiff first decides whether or not to sue the defendant, by filing a case and paying a non-refundable filing fee of \$100. If he does not sue, the game ends and each gets 0. If he sues, then he is to decide whether or not to offer a settlement of amount $\$25,000$. If he offers a settlement, then the Defendant either accepts the offer, in which case the Defendant pays the settlement amount to the Plaintiff, or rejects the offer. If the Defendant rejects the offer, or the Plaintiff does not offer a settlement, the Plaintiff can either pursue the suit and go to court, or drop the suit. Applying backward induction, find a Nash equilibrium.

(b) Now imagine that the Plaintiff has already paid his lawyer $20,000 for the legal costs, and the lawyer is to keep the money if they do not go to court. That is, independent of whether or not they go to court, the Plaintiff pays the $20,000 of legal costs. Applying backward induction, find a Nash equilibrium. under the new scenario.

3. [Homework 2, 2006] This question is about a tv game, called Deal or No Deal. There are two players: Banker and Contestant. There are n cash prizes, $v_1, \ldots, v_n$, which are randomly put in n cases, 1, ..., n. Each permutation is equally likely. Neither player knows which prize is in which case. The contestant owns Case 1. There are $n-1$ periods. At each period, Banker makes a cash offer p. The Contestant is to accept ("Deal") or reject ("No Deal") the offer. If she accepts the offer, the Banker buys the case from the Contestant at price p and the game ends. (Banker gets the prize in Case 1 minus p, and the Contestant gets p.) If she rejects the offer, then one of the remaining cases is opened to reveal its content to the players, and we proceed to the next period. When all the cases 2,..., n are opened, the game automatically ends; the Banker gets 0 and the Contestant gets the prize in Case 1. Assume that the utility of having x dollar is x for the Banker and $\sqrt{x}$ for the Contestant. Everything described is common knowledge.

(a) Apply backward induction to find an equilibrium of this game. (Assume that the Contestant accepts the offer whenever she is indifferent between accepting or rejecting the offer. Solving the special case in part b first may be helpful.)

(b) What would be your answer if $n = 3$, $v_1 = 1$, $v_2 = 100$, and $v_3 = 10000$.

4. [Midterm 1, 2007] [**Read the bonus note at the end before you answer the question**.] This question is about arbitration, a common dispute resolution method in the US. We have a Worker, an Employer, and an Arbitrator. They

want to set the wage w. If they determine the wage w at date t, the payoffs of the Worker, the Employer and the Arbitrator will be $\delta^t w$, $\delta^t (1-w)$ and $w(1-w)$, respectively, where $\delta \in (0,1)$. The timeline is as follows:

- At $t = 0$,
  - the Worker offers a wage w_0;
  - the Employer accepts or rejects the offer;
  - if she accepts the offer, then the wage is set at w_0 and the game ends; otherwise we proceed to the next date;
- at $t = 1$,
  - the Employer offers a wage w_1;
  - the Worker accepts or rejects the offer;
  - if he accepts the offer, then the wage is set at w_1 and the game ends; otherwise we proceed to the next date;
- at $t = 2$, the Arbitrator sets a wage $w_2 \in [0,1]$ and the game ends.

Compute an equilibrium of this game using backward induction.

Bonus: If you solve the following variation instead, then you will get extra 10 points (45 points instead of 35 points). *Final Offer Arbitration:* At $t = 2$, the Arbitrator sets a wage $w_2 \in \{w_0, w_1\}$, i.e., the Arbitrator has to choose one of the offers made by the parties.

14.12 Economic Applications of Game Theory
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